

A NUMERICAL SCHEME FOR IMPACT PROBLEMS

LAETITIA PAOLI * AND MICHELLE SCHATZMAN †

Abstract. We consider a mechanical system with impact and n degrees of freedom, written in generalized coordinates. The system is not necessarily Lagrangian. The representative point is subject to a constraint: it must stay inside a closed set K with boundary of class C^3 . We assume that, at impact, the tangential component of the impulsion is conserved, while its normal coordinate is reflected and multiplied by a given coefficient of restitution $e \in [0, 1]$: the mechanically relevant notion of orthogonality is defined in terms of the local metric for the impulsions (local cotangent metric). We define a numerical scheme which enables us to approximate the solutions of the Cauchy problem: this is an *ad hoc* scheme which does not require a systematic search for the times of impact. We prove the convergence of this numerical scheme to a solution, which yields also an existence result. Without any *a priori* estimates, the convergence and the existence are local; with some *a priori* estimates, the convergence and the existence are proved on intervals depending exclusively on these estimates. The technique of proof uses a localization of the scheme close to the boundary of K ; this idea is classical for a differential system studied in the framework of flows of a vector field; it is much more difficult to implement here, because finite differences schemes are only approximately local: straightening the boundary creates quadratic terms which cause all the difficulties of the proof.

Key words. Impact, coefficient of restitution, numerical scheme, convergence, local existence, global existence.

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1. Introduction. We study in this article a numerical approximation of dynamics with impact with a finite number of degrees of freedom and a smooth constraint.

The set of constraints is denoted K and satisfies the following assumptions:

$$K \text{ is a closed subset of } \mathbb{R}^d \text{ with non empty interior;} \quad (1.1a)$$

$$\begin{cases} \text{the boundary } \partial K \text{ of } K \text{ is an embedded sub-manifold} \\ \text{of class } C^3 \text{ of } \mathbb{R}^d; \end{cases} \quad (1.1b)$$

$$K \text{ lies on only one side of } \partial K. \quad (1.1c)$$

It is possible to find a function ϕ of class C^3 such that

$$K = \{u \in \mathbb{R}^d : \phi(u) \geq 0\}$$

and the differential $d\phi$ does not vanish on $\partial K = \{u \in \mathbb{R}^d : \phi(u) = 0\}$.

Let f be a continuous function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}^d which is locally Lipschitz continuous with respect to its last two arguments, and let $M(u)$ be the mass matrix: $u \mapsto M(u)$ is a mapping of class C^3 from \mathbb{R}^d to the set of symmetric positive definite matrices.

The free dynamics of the system are written in generalized coordinates as

$$M(u)\ddot{u} = f(\cdot, u, p), \quad p = M(u)\dot{u}. \quad (1.2)$$

*UMR 5585 CNRS Analyse Numérique, Faculté des Sciences, Université Jean Monnet, 23 Rue du Docteur Paul Michelon, 42023 Saint-Etienne Cedex 2, France, (paoli@anumsun1.univ-st-etienne.fr)

†UMR 5585 CNRS Analyse Numérique, Université Lyon 1, 69622 Villeurbanne Cedex, France(schatz@maply.univ-lyon1.fr)

This system is more general than the system obtained in Lagrangian mechanics, since we want to include possible dissipative terms in the dynamics of the problem under discussion.

Let us give the few geometric notations which are absolutely necessary here, since we use a Riemannian metric; the cotangent bundle $T^*\mathbb{R}^d$ is identified to $\mathbb{R}^d \times \mathbb{R}^d$, and its elements are denoted as pairs (u, ξ) ; at each point u of \mathbb{R}^d the metric tensor for tangent vectors is defined by the matrix $M(u)$, and the metric tensor for cotangent vectors is defined by the matrix $M(u)^{-1}$. The scalar product of two vectors x and y in the tangent space at u is denoted by $\langle x, y \rangle_u$; coordinate-wise it can be expressed as $x^T M(u) y$ where x and y are column vectors. The scalar product of two vectors ξ and η in the cotangent space at u is denoted by $\langle \xi, \eta \rangle_u^*$ and coordinate-wise it is equal to $\xi^T M(u)^{-1} \eta$. The corresponding norms of vectors and covectors are denoted respectively by $|x|_u$ and $|\xi|_u^*$.

Therefore, a cotangent vector (u, ξ) belonging to $T^*\mathbb{R}^d$ is orthogonal to the cotangent vector (u, η) iff $\langle \xi, \eta \rangle_u^*$ vanishes.

With these notations, if the velocity of the system is \dot{u} , the generalized impulsion is $M(u)\dot{u} = p$ and (u, p) belongs to the cotangent space $T^*\mathbb{R}^d$. Whenever we take the orthogonal of a vector or a vector subspace of the tangent or the cotangent space at u , we always use the relevant metric tensor; therefore it is important to know which of the vectors under consideration are cotangent and which are tangent. Of course, all the differential forms are cotangent vectors.

Let us describe now the system satisfied by the problem with impact: we replace (1.2) by

$$M(u)\ddot{u} = \mu + f(\cdot, u, p), \quad (1.3)$$

and since we cannot expect to have global solutions in general, μ is an unknown measure on $[t_0, t_0 + \bar{\tau}]$ with values in \mathbb{R}^d which describes the reaction of the constraints: μ has the following properties: if $d\phi$ denotes the differential of ϕ , then

$$\text{supp}(\mu) \subset \{t \in [t_0, t_0 + \bar{\tau}] : \phi(u(t)) = 0\}, \quad (1.4a)$$

$$\mu = \lambda d\phi(u), \quad (1.4b)$$

$$\lambda \geq 0 \text{ almost everywhere on } [t_0, t_0 + \bar{\tau}]. \quad (1.4c)$$

We require the following functional properties for u :

$$\begin{cases} u \text{ is a continuous function taking its values in } K \\ \text{for all } t \in [t_0, t_0 + \bar{\tau}], \end{cases} \quad (1.5a)$$

$$\dot{u} \text{ is of bounded variation over } [t_0, t_0 + \bar{\tau}]. \quad (1.5b)$$

If \dot{u} is of bounded variation, p is also of bounded variation. Assume that $u(t)$ belongs to ∂K ; we decompose $p(t-0)$ and $p(t+0)$ on $\mathbb{R}d\phi(u(t)) \oplus d\phi(u(t))^\perp$; here the \perp sign means the orthogonality with respect to the local cotangent metric. We integrate (1.3) on a small neighborhood of t , relation (1.4b) implies that the component of $p(t-0)$ on $d\phi(u(t))^\perp$ is conserved.

Therefore, we have to make a supplementary assumption in order to have a complete description of the impact; we choose a constitutive law of the impact using a coefficient of restitution: thus we will assume that there exists $e \in [0, 1]$ such that the component of $\dot{p}(t+0)$ along $\mathbb{R}d\phi(u)$ is equal to $-e$ times the component of $p(t-0)$

on $\mathbb{R}d\phi(u)$. In other words, we have

$$p(t+0) = p(t-0) - (1+e) \frac{\langle d\phi(u(t)), p(t-0) \rangle_{u(t)}^*}{\langle d\phi(u(t)), d\phi(u(t)) \rangle_{u(t)}^*} d\phi(u(t)). \quad (1.6)$$

The set of admissible initial data \mathbb{D} will be

$$\mathbb{D} = \left\{ (t_0, u_0, p_0) \in [0, T) \times K \times \mathbb{R}^d : \right. \\ \left. \text{if } u_0 \in \partial K, \text{ then } \langle p_0, d\phi(u_0) \rangle_{u_0}^* \geq 0 \right\}. \quad (1.7)$$

This choice is equivalent to the convention that there is no impact at the initial time t_0 .

Given initial conditions $(t_0, u_0, p_0) \in \mathbb{D}$, we require that the following Cauchy data be satisfied:

$$u(t_0) = u_0, \quad (1.8)$$

and

$$p(t_0) = p_0. \quad (1.9)$$

For all initial data $(t_0, u_0, p_0) \in \mathbb{D}$ we will obtain the existence of a local solution to (1.3), (1.4a), (1.4b), (1.4c) and (1.6) belonging to the functional class defined by (1.5a) and (1.5b) and satisfying the initial conditions (1.8) and (1.9).

The existence of this local solution is obtained by defining a numerical scheme, whose convergence will be shown in appropriate functional spaces; the limit of the approximation will be a solution of our problem.

The distance on \mathbb{R}^d is defined with the help of the Riemannian metric: if $s \mapsto u(s)$ is a C^1 mapping from $[a, b]$ to \mathbb{R}^d , the Riemannian length of the image of u is

$$\ell(u) = \int_a^b |\dot{u}(s)|_{u(s)} ds.$$

This curve length is invariant by a diffeomorphic change of parameter. Therefore, we may assume that $a = 0$ and $b = 1$. The distance from x to y is the lower bound of the length of the curves from x to y , or in other words:

$$\text{dist}(x, y) = \inf \{ \ell(u) : u \in C^1([0, 1]), \quad u(0) = x, \quad u(1) = y \}.$$

It is classical that the lower bound is attained on the geodesics for the given Riemannian metric; it is also known that for each point x there exists $r > 0$ such that if $\text{dist}(x, y) \leq r$ there is only one geodesic from x to y .

We denote by $\text{dist}(x, E)$ the Riemannian distance of a point x to a set E .

Under assumptions (1.1), a projection on ∂K can be defined uniquely on an appropriate neighborhood of ∂K ; more precisely, for all compact $\mathcal{C} \subset \partial K$, there exists a neighborhood of \mathcal{C} on which the projection $P_{\partial K}$ is uniquely defined, and there exists a unique geodesic joining a point of this neighborhood to its projection. This projection $P_{\partial K}$ is characterized by the relation

$$\forall y \in \partial K, \quad \text{dist}(P_{\partial K}x, x) \leq \text{dist}(y, x). \quad (1.10)$$

This projection is of class C^2 .

For all x in ∂K , denote by $N(x)$ the interior unit normal vector: this means that $|N(x)|_x$ is equal to 1 and that it is orthogonal to the tangent space at $P_{\partial K}x$ with respect to the scalar product in the tangent space, i.e. for all y such that $d\phi(x)y$ vanishes, $\langle y, N(x) \rangle_x = 0$. The smoothness of ∂K implies that the mapping $z \mapsto N(z)$ is of class C^2 .

When the geodesic from x to $P_{\partial K}x$ is unique it is tangent at $P_{\partial K}x$ to $N(P_{\partial K}x)$.

Starting from this projection on ∂K , we can define a projection on K as follows: for each compact \mathcal{C} included in K , there exists a relatively compact neighborhood \mathcal{U} of \mathcal{C} on which P_K is defined by

$$P_K(x) = \begin{cases} P_{\partial K}(x) & \text{if } x \notin K \\ x & \text{otherwise.} \end{cases} \quad (1.11)$$

The reader will check that P_K is Lipschitz continuous over \mathcal{U} and that P_Kx realizes the minimum of the distance from x to K .

Given two positive numbers $h^* \leq 1$ and T , assume that F is a continuous function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times [0, h^*]$ to \mathbb{R}^d , which is locally Lipschitz continuous with respect to its second, third and fourth arguments; assume moreover that F is consistent with f , i.e. that for all $t \in [0, T]$, for all u and v in \mathbb{R}^d

$$F(t, u, u, v, 0) = M(u)^{-1}f(t, u, M(u)v). \quad (1.12)$$

We approximate the solution of (1.3), (1.4a), (1.4b), (1.4c), (1.5a), (1.5b), (1.8), (1.9) by the following numerical scheme: the initial values U^0 and U^1 are given by the initial position

$$U^0 = u_0, \quad (1.13)$$

and the position at the first time step

$$U^1 = u_0 + hM(u_0)^{-1}p_0 + hz(h), \quad (1.14)$$

where $z(h)$ tends to 0 as h tends to 0.

We will use systematically henceforth the notation

$$t_m = t_0 + mh. \quad (1.15)$$

Given U^{m-1} and U^m , U^{m+1} is defined by the relations

$$U^{m+1} = -eU^{m-1} + (1+e)P_K \left(\frac{2U^m - (1-e)U^{m-1} + h^2F^m}{1+e} \right) \quad (1.16)$$

and

$$F^m = F \left(t_m, U^m, U^{m-1}, \frac{U^{m+1} - U^{m-1}}{2h}, h \right) \quad (1.17)$$

provided that U^{m+1} is unique in a neighborhood of U^m .

A commentary on the construction of this scheme from the point of view of convex analysis will be useful here. We refer to the book of Rockafellar [28] for more information on the basic ideas in convex analysis to be used below.

Let us assume provisionally that the set of constraints K is convex and that the mass matrix is equal to the identity matrix on \mathbb{R}^d . Then the Riemannian structure of \mathbb{R}^d is simply its Euclidean structure.

Recall that the indicator function ψ_K of a closed convex set K is defined by

$$\psi_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{otherwise,} \end{cases} \quad (1.18)$$

and its sub-differential $\partial\psi_K$ is a function from K to the set of closed convex sets given by

$$\partial\psi_K(x) = \begin{cases} \{0\} & \text{if } x \in \text{int}(K), \\ \mathbb{R}^+ N(x) & \text{if } x \in \partial K. \end{cases} \quad (1.19)$$

For all $\lambda > 0$, the multivalued equation

$$x + \lambda \partial\psi_K(x) \ni f \quad (1.20)$$

has a unique solution given by

$$x = P_K(f), \quad (1.21)$$

where P_K is the usual projection on the closed convex set K in Euclidean \mathbb{R}^d .

In the announcement [21], where we assumed that the set of constraints K was convex and the geometry was Euclidean, we had defined the numerical scheme by the multivalued equation

$$\frac{U^{m+1} - 2U^m + U^{m-1}}{h^2} + \partial\psi_K\left(\frac{U^{m+1} + eU^{m-1}}{1+e}\right) \ni F^m. \quad (1.22)$$

We may rewrite (1.22) as

$$\begin{aligned} & \frac{U^{m+1} + eU^{m-1}}{1+e} + \frac{h^2}{1+e} \partial\psi_K\left(\frac{U^{m+1} + eU^{m-1}}{1+e}\right) \\ & \ni \frac{2U^m - (1-e)U^{m-1} + h^2 F^m}{1+e}, \end{aligned} \quad (1.23)$$

which reduces, thanks to (1.20) and (1.21) to relation (1.16).

If we generalize (1.22) to a non convex K with a general mass matrix, we cannot use the apparatus of convex analysis, and there is no good reason to use the even more technical apparatus of non-convex analysis *à la* Clarke: this theory is useful when the corners of K are not convex; in the mechanical setting, corners are convex, since they appear as the intersection of smooth sets of constraints. Here, the problem is even simpler because we do not have any corners.

The boundary ∂K is smooth, and as we expect that for small h , the U^m 's will stay close to K , we still have a projection of $(2U^m - (1-e)U^{m-1} + h^2 F^m)/(1+e)$ on K , and thus we start from (1.16) to define the numerical scheme.

The original definition reappears as follows: define

$$W^m = \frac{2U^m - (1-e)U^{m-1} + h^2 F^m}{1+e}, \quad (1.24)$$

that will be used in many places in the upcoming proofs. With this definition, (1.16) is rewritten as

$$U^{m+1} = -eU^{m-1} + (1+e)P_K(W^m).$$

Hence, if we define

$$Z^m = \frac{U^{m+1} + eU^{m-1}}{1+e}, \quad (1.25)$$

we find that

$$Z^m = P_K(W^m). \quad (1.26)$$

If we subtract (1.25) from (1.24), we can see that

$$\frac{U^{m+1} - 2U^m + U^{m-1}}{h^2} + \frac{1+e}{h^2}(W^m - Z^m) = F^m \quad (1.27)$$

which reduces to (1.22) in the convex case with a trivial mass matrix.

Another way of writing (1.27) is to define the discrete velocity V^m by

$$V^m = \frac{U^{m+1} - U^m}{h}. \quad (1.28)$$

Then, (1.27) can be rewritten as

$$V^m - V^{m-1} - hF^m = \frac{(1+e)(Z^m - W^m)}{h}. \quad (1.29)$$

A strict contraction argument in \mathbb{R}^d gives the existence of a unique U^m for small values of m and h . As the projection on K is uniquely defined only in a neighborhood of K , and is only Lipschitz continuous, the iteration of a fixed point argument might request smaller and smaller bounds on the time step h , and there is no guarantee that we could integrate numerically on a time interval bounded from below, for any initial time step size.

It should be noted that this difficulty is specific to the non convex case.

Let us outline now the structure of the article and of the proofs. In the one-dimensional case, the main estimates are given by lemma 2.1, in section 2. In section 3, we will straighten the boundary, a natural geometrical idea.

While the system (1.3)–(1.6) is nicely transformed under a diffeomorphism, the numerical scheme (1.13), (1.14), (1.16) and (1.17) does not behave well under diffeomorphism. The reason is that a numerical scheme is not a local object: when we define a discrete velocity by subtracting U^m from U^{m+1} , we use locally a vector structure which is not intrinsic from the point of view of differential geometry. In particular, if we apply a diffeomorphism to the numerical scheme, we will find another numerical scheme which will look much more complicated than the previous one, since it will contain a number of small term which show the lack of an intrinsic description of the scheme. After a very technical proof, we find two constants C_3 and τ such that for initial data in a compact subset of the admissible set, and for all small enough h and all $m \leq \tau/h$, the discrete velocity is bounded:

$$\sup |V^m| \leq C_3.$$

Since uniqueness is not true in general [2], [29], and hypotheses of analyticity are often but not always used for the proof of uniqueness [25], [27], [30], [5], [1], the proof of convergence of the numerical approximation is delicate also for this reason.

However, there is a bonus: all the effort made to prove the local convergence of the numerical scheme provides us with a local existence proof for our problem. In sections 4, 5, 6 and 7, we prove estimates on the discrete acceleration, we establish the variational properties of the limit of the numerical scheme, and we study the transmission of energy at impact, as well as the passage to the limit for the initial conditions. All these results are obtained under the assumption that on a certain time interval starting at t_0 , the discrete velocity is bounded independently of the time step.

As a preliminary to the global existence proof, we give *a priori* estimates on problem (1.3)–(1.9) in section 8, which is completely independent from the remainder of the article.

In section 9, we establish a very weak semi-continuity for the supremum of the local norm of the discrete velocities; this result enables us to obtain a global existence and convergence theorem.

This article is of a theoretical nature: the existence result obtained here is a generalization of [29], [4], [26], [19], [20].

The numerical scheme analyzed here has been implemented in the case of a trivial mass matrix in [19], [22], [18], [23]. In all these articles, we compared the performances of this scheme with those of a method based on the detection of impact. When the impact times are isolated, the scheme by detection of impacts is more precise than the present scheme. As soon as the restitution coefficient is strictly less than one, we find systematically non-isolated impact times. In all cases, the present scheme is substantially faster. Since the phenomena that we want to approximate are highly nonlinear and often very sensitive to the initial data, the issue of precision is not necessarily crucial. Our numerical experiments show that the performance of the present numerical scheme is quite satisfactory from the point of view of qualitative conclusions.

The case of a non-trivial mass matrix, and a stiff system, indeed the case of the discretization of a beam has been addressed in [24].

Let us remark that many articles have been devoted to the problem treated here, under the assumption of anelasticity, i.e. a situation where the normal component of the impulsion vanishes after the impact; Moreau applied Gauss' principle of least constraint to unilateral problems in order to justify his choice of anelastic impact [12], which eventually led him to sweeping processes [15], followed by [13], [14]; dry friction enters in Moreau's work as [16]; frictionless anelastic impact starts as [17], and the mathematical theory is tackled by M. Monteiro-Marques in a series of articles: his main contributions are [10] for the general theory of differential inclusions, [11] for one-dimensional dynamics with friction, [8] which adds percussion to the previous framework; this work is improved as [9], where dynamics of n particles on a plane with normal friction are considered. The discretization approach has been taken up by Monteiro-Marques and Kuntze in [7], but most significantly by Stewart and Trinkle: they use that approach in [31], [33] and [34]; the real coronation is the beautiful and difficult article of Stewart [32], which concludes the study of dynamics with friction and anelastic impact for a finite number of degrees of freedom, and one constraint, and still important results in the multiple constraint case.

The philosophy of this long list of works is somewhat different from ours: we feel that not all impacts are anelastic, and we were originally motivated by continuous

media; thus, we wanted to develop methods which work well for stiff systems of ordinary differential equations. From this point of view, any method which has to calculate with some precision the impact times is doomed to failure. On the other hand, the precision of the method presented here needs improvement, and globally, it would make sense to agree on benchmarks which would enable the end-user to decide between different numerical methods.

2. The heart of the estimates. In the one-dimensional case, the main estimate on the numerical scheme is described in the following lemma; we recall the definition

$$r^+ = \max(r, 0).$$

LEMMA 2.1. *Let the real-valued sequence $(y^m)_m$ satisfy the following recurrence relation for all $m \geq 1$:*

$$y^{m+1} = -ey^{m-1} + (2y^m - (1-e)y^{m-1})^+ + h^2\lambda^m. \quad (2.1)$$

Then, for all $m \geq 2$, the discrete velocity

$$\eta^m = (y^{m+1} - y^m)/h \quad (2.2)$$

satisfies the estimate

$$|\eta^m| \leq \max(|\eta^{m-1}|, e|\eta^{m-2}|) + h|\lambda^m| + h|\lambda^{m-1}|. \quad (2.3)$$

Proof. Assume first that $2y^m - (1-e)y^{m-1}$ is non negative, and substitute $y^{m+1} = y^m + h\eta^m$, $y^{m-1} = y^m - h\eta^{m-1}$ into (2.1); we obtain

$$\eta^m = \eta^{m-1} + h\lambda^m,$$

so that

$$|\eta^m| \leq |\eta^{m-1}| + h|\lambda^m|. \quad (2.4)$$

Assume now that $2y^m - (1-e)y^{m-1}$ is strictly negative. On one hand, (2.1) implies the relation

$$\eta^m = e\eta^{m-1} - \frac{1+e}{h}y^m + h\lambda^m;$$

the assumption on the sign of $2y^m - (1-e)y^{m-1}$ is equivalent to

$$\frac{(1+e)y^m}{h} < -(1-e)\eta^{m-1},$$

and therefore

$$\eta^m > \eta^{m-1} + h\lambda^m. \quad (2.5)$$

On the other hand, we subtract from the relation

$$y^{m+1} + ey^{m-1} = h^2\lambda^m$$

the inequality implied by (2.1) with m substituted by $m - 1$:

$$y^m + ey^{m-2} \geq h^2 \lambda^{m-1},$$

and we infer that

$$\eta^m \leq -e\eta^{m-2} + h(\lambda^m - \lambda^{m-1}). \quad (2.6)$$

When we summarize (2.4), (2.5) and (2.6), we find (2.3). \square

Later on, we will give a d -dimensional version of (2.3), where the main difference is due to geometric effects: there will be a term resembling λ^m , and the game will be to prove a bound on this term.

3. Existence of $(U^m)_{0 \leq m \leq \lfloor \tau/h \rfloor}$ for some $\tau > 0$. We use systematically the floor and ceiling notations: when r is a real number, the floor $\lfloor r \rfloor$ of r is the largest integer at most equal to r , and the ceiling $\lceil r \rceil$ is the smallest integer at least equal to r .

The main result of this section is the existence of a number $\tau > 0$ such that for all small enough h and all $m \leq \lfloor \tau/h \rfloor$ there exists indeed a discrete solution of (1.16) and (1.17), whose discrete velocity is bounded independently of h . In fact, we prove a stronger result: provided that the first two discrete velocities are bounded, we find a uniform lower bound on τ when the initial position belongs to a compact subset of K .

We prove first the existence of U^2 under appropriate assumptions on U^0 and U^1 . This proof decomposes in two lemmas: the first lemma is strictly an initial condition statement, in which no uniformity with respect to initial conditions can be obtained. The second one will be used in the foregoing induction proofs.

LEMMA 3.1. *For all $(t_0, u_0, M(u_0)v_0) \in \mathbb{D}$, for all U^1 satisfying (1.14), and for all small enough h , there exists a solution U^2 of (1.16) for $m = 2$ satisfying*

$$|U^2 - U^1|_{u_0} \leq 2|v_0|_{u_0} h.$$

Proof. Let $r > 0$ be such that P_K is Lipschitz continuous on

$$B_{u_0}(u_0, r) = \{u \in \mathbb{R}^d : |u - u_0|_{u_0} \leq r\}.$$

Define \tilde{C}_1 by

$$\begin{aligned} \tilde{C}_1 = \max \{ & |F(t, u, u', 0, h)|_{u_0} : t \in [0, T], |u - u_0|_{u_0} \leq r, \\ & |u' - u_0|_{u_0} \leq r, h \in [0, h^*] \}, \end{aligned}$$

and let \tilde{L} be the Lipschitz constant defined by

$$\begin{aligned} \tilde{L} = \sup \Big\{ & \frac{|F(t, u, u', v, h) - F(t, u, u', v', h)|_{u_0}}{|v - v'|_{u_0}} : |u - u_0|_{u_0} \leq r, \\ & |u' - u_0|_{u_0} \leq r, |v|_{u_0} \leq 2|v_0|_{u_0} + 1, |v'|_{u_0} \leq 2|v_0|_{u_0} + 1, \\ & v \neq v', h \in [0, h^*] \Big\}. \end{aligned}$$

Finally, let $\tilde{\gamma}$ be the Lipschitz constant of P_K defined by

$$\tilde{\gamma} = \sup \left\{ \frac{|P_K u - P_K u'|_{u_0}}{|u - u'|_{u_0}} : |u - u_0|_{u_0} \leq r, |u' - u_0|_{u_0} \leq r, u \neq u' \right\}.$$

There exists a function $\hat{z}(t)$ which is bounded in a neighborhood of 0 such that for small positive values of t :

$$P_K(u_0 + tv_0) = u_0 + tv_0 + t^2 \hat{z}(t); \quad (3.1)$$

indeed, if v_0 vanishes, or if u_0 belongs to $\text{int}(K)$, or if u_0 belongs to ∂K and the scalar product $\langle v_0, N(u_0) \rangle_{u_0}$ is strictly positive, \hat{z} vanishes; if u_0 belongs to ∂K and $\langle v_0, N(u_0) \rangle_{u_0}$ vanishes, while v_0 does not vanish, (3.1) is a consequence of the smoothness of $P_{\partial K}$ in a neighborhood of u_0 : for the values of t for which $u_0 + tv_0$ belongs to K , \hat{z} vanishes; for the values of t for which $u_0 + tv_0$ does not belong to K , a Taylor expansion shows that

$$P_{\partial K}(u_0 + tv_0) = u_0 + tv_0 + O(t^2),$$

hence (3.1). With the change of variable $U^2 = U^1 + tV^1$, equation (1.16) is equivalent to

$$v = \tilde{G}(v)$$

where the function \tilde{G} is defined by

$$\begin{aligned} \tilde{G}(v) = & -V^0 + \frac{1+e}{h} \left[P_K \left(U^0 + \frac{2h}{1+e} V^0 \right. \right. \\ & \left. \left. + \frac{h^2}{1+e} F \left(t_1, U^1, U^0, \frac{V^0 + v}{2}, h \right) \right) - U^0 \right]. \end{aligned}$$

Let us check that \tilde{G} is a strict contraction on $B_{u_0}(0, 2|v_0|_{u_0} + 1)$: if $|v|_{u_0} \leq 2|v_0|_{u_0}$, then

$$\left| \frac{v + V^0}{2} \right| \leq |v_0|_{u_0} + \frac{1}{2} |v_0 + z(h)|_{u_0} + \frac{1}{2};$$

therefore, for h small enough, $|(v + V_0)/2|_{u_0}$ is at most equal to $2|v_0|_{u_0}$, and we can use the definitions of \tilde{L} and \tilde{C}_1 :

$$\left| \left(F(t_1, U^1, U^0, \frac{V^0 + v}{2}, h) \right) \right|_{u_0} \leq \tilde{C}_1 + \tilde{L}(2|v_0|_{u_0} + 1). \quad (3.2)$$

We estimate $G(v)$ as follows: by the triangle inequality, and the Lipschitz condition on P_K ,

$$\begin{aligned} |G(v)|_{u_0} \leq & \frac{1+e}{h} \tilde{\gamma} \left| U^0 + \frac{2h(V^0 - v_0)}{1+e} + \frac{h^2}{1+e} F - u_0 \right|_{u_0} \\ & + \left| -V^0 + \frac{1+e}{h} \left(P_K \left[u_0 + \frac{2hv_0}{1+e} \right] - u_0 \right) \right|_{u_0}. \end{aligned}$$

We apply (3.1), (1.14) and (3.2), and we find

$$\begin{aligned} |G(v)|_{u_0} &\leq \tilde{\gamma} \left[2|z(h)|_{u_0} + h(\tilde{C}_1 + \tilde{L}(2|v_0|_{u_0} + 1)) \right] \\ &\quad + |v_0|_{u_0} + |z(h)|_{u_0} + \frac{4h}{1+e} \left| \hat{z} \left(\frac{2h}{1+e} \right) \right|_{u_0}. \end{aligned}$$

Therefore, for h small enough, G maps $B_{u_0}(0, 2|v_0|_{u_0} + 1)$ to itself. Moreover, the Lipschitz constant of G on this ball is at most equal to $\tilde{\gamma}\tilde{L}h/2$. This proves that G has a fixed point in $B_{u_0}(0, 2|v_0|_{u_0} + 1)$ for small enough values of h and completes the proof of the lemma. \square

Here is the statement of the uniformizable estimate, which will be used throughout the induction. We will say that U^0 and U^1 satisfy condition $E(\bar{u}, r_0, C_2, h)$ if

$$|U^0 - \bar{u}| \leq r_0, \quad |U^1 - \bar{u}| \leq r_0, \quad |U^1 - U^0| \leq C_2 h \quad (3.3)$$

and moreover U^2 is uniquely defined in $B(\bar{u}, r_0)$ by

$$\frac{U^2 + eU^0}{1+e} = P_K \left(\frac{2U^1 - (1-e)U^0 + h^2 F^1}{1+e} \right), \quad (3.4)$$

and the following inequalities are satisfied:

$$|U^2 - \bar{u}| \leq r_0, \quad |U^2 - U^1| \leq C_2 h. \quad (3.5)$$

LEMMA 3.2. *For all $\bar{u} \in K$, there exists r_0 such that for all $C_2 > 0$ it is possible to find $h_1 > 0$ and $C'_2 < \infty$ with the following properties: for all $h \leq h_1$ and for all choice of U^0, U^1 satisfying condition $E(\bar{u}, r_0, C_2, h)$ i.e. (3.3), (3.4) and (3.5), there exists a unique U^3 satisfying (1.16) for $m = 2$ and the estimate*

$$|U^3 - U^2| \leq C'_2 h.$$

Proof. Subtract (1.16) for $m = 1$ from (1.16) for $m = 2$; with the change of variable $U^3 = U^2 + hV^2$, we have to show the existence of a solution of

$$\begin{aligned} V^2 &= -eV^0 + \frac{1+e}{h} P_K \left(\frac{2U^1 + 2hV^1 - (1-e)(U^0 + hV^0) + h^2 F^2}{1+e} \right) \\ &\quad - \frac{1+e}{h} P_K \left(\frac{2U^1 - (1-e)U^0 + h^2 F^1}{1+e} \right). \end{aligned}$$

If we denote by $G(V^2)$ the right hand side of the above equation, we have to choose a parameter C'_2 such that G will be a strict contraction of the ball $B(0, C'_2)$ into itself. Let r_0 be such that P_K is Lipschitz continuous on $B(\bar{u}, 2r_0)$; denote by γ the Lipschitz constant of P_K over this ball, and define

$$C'_2 = (3\gamma + 1)C_2; \quad (3.6)$$

let C_1 be given by

$$\begin{aligned} C_1 &= \sup \{ |F(t, u, u', 0, h)| : t \in [0, T], |u - \bar{u}| \leq 2r_0, \\ &\quad |u' - \bar{u}| \leq 2r_0, h \in [0, h^*] \}. \end{aligned} \quad (3.7)$$

Denote finally by L the Lipschitz constant of F defined as follows:

$$L = \sup \left\{ \frac{|F(t, u, u', v, h) - F(t, u, u', v', h)|}{|v - v'|} : t \in [0, T], |u - \bar{u}| \leq 2r_0, \right. \\ \left. |u' - \bar{u}| \leq 2r_0, |v| \leq C'_2, |v'| \leq C'_2, h \in [0, h^*], v \neq v' \right\}. \quad (3.8)$$

Then, we have the estimate for $|v| \leq C'_2$:

$$|F(t_2, U^1, U^2, (v + V^1)/2, h)| \leq C_1 + LC'_2, \\ |F(t_1, U^0, U^1, (V^1 + V^0)/2, h)| \leq C_1 + LC'_2.$$

It is straightforward that

$$\max \left(\left| \frac{2U^1 - (1 - e)U^0 + h^2 F^1}{1 + e} - \bar{u} \right|, \right. \\ \left. \left| \frac{2U^2 - (1 - e)U^1 + h^2 F^2}{1 + e} - \bar{u} \right| \right) \\ \leq r_0 + \frac{2hC'_2}{1 + e} + \frac{h^2(C_1 + LC'_2)}{1 + e}.$$

Therefore, if h_1 satisfies the estimate

$$\frac{2h_1 C'_2}{1 + e} + \frac{h_1^2 (C_1 + LC'_2)}{1 + e} \leq r_0, \quad (3.9)$$

we may use the Lipschitz continuity of P_K on the ball of radius $2r_0$ about \bar{u} , and we find that if v belongs to $B(0, C'_2)$,

$$|G(v)| \leq eC_2 + \gamma((3 - e)C_2 + 2h(C_1 + LC'_2));$$

We observe that γ is at least equal to 1, since $DP_{\partial K}$ has eigenvectors relative to the eigenvalue 1; therefore

$$e + (3 - e)\gamma < 3\gamma + 1;$$

thus, if h is so small that

$$[e + (3 - e)\gamma]C_2 + 2\gamma h_1(C_1 + LC'_2) \leq (3\gamma + 1)C_2 = C'_2, \quad (3.10)$$

G maps $B(0, C'_2)$ into itself; moreover, the Lipschitz constant of G over this ball is at most equal to $\gamma Lh/2$; if

$$\gamma Lh_1 < 2, \quad (3.11)$$

G is a strict contraction from $B(0, C'_2)$ to itself, which proves the lemma. \square

When \bar{u} belongs to ∂K , we need local coordinates in which the projection P_K is particularly simple. They are defined in the following fashion: we choose a coordinate frame in \mathbb{R}^d such that

- $\bar{u} = 0$;
- the tangent hyperplane to ∂K at 0 is the hyperplane of the first $d - 1$ coordinates;

- the positive direction of the d -th coordinate axis points inside K .
- For a d -dimensional vector x , we will use the notation

$$x' = (x_1, \dots, x_{d-1}).$$

Locally, ∂K is a graph over the hyperplane of the first $d - 1$ coordinates, and it can be parameterized as

$$\chi(x') = \begin{pmatrix} x' \\ H(x') \end{pmatrix},$$

where x' belongs to \mathbb{R}^{d-1} , H is of class C^3 and $DH(0)$ vanishes. Let $s \mapsto \psi(s, z)$ be the parameterization of the geodesic starting at $z \in \partial K$ with an initial velocity equal to $-N(z)$ which satisfies

$$\left| \frac{\partial \psi}{\partial s}(s, z) \right|_{\psi(s, z)} = 1. \quad (3.12)$$

Let Ψ be defined by

$$\Psi(x', y) = \psi(-y, \chi(x')); \quad (3.13)$$

the function Ψ is of class C^2 in a neighborhood of 0; its derivative at 0 has the block representation

$$D\Psi(0, 0) = \begin{pmatrix} \mathbf{1}^{d-1} & | & N(0) \\ 0 & & \end{pmatrix}; \quad (3.14)$$

it is invertible, since $N(0)$ does not belong to the tangent plane at 0 to ∂K . Thus Ψ is a local diffeomorphism from a neighborhood \mathcal{U} of 0 to a neighborhood $\Psi(\mathcal{U})$ of 0. In particular, we may assume that \mathcal{U} contains a compact neighborhood of 0 of the form $\overline{\mathcal{O}} \times [-r_1, r_1]$ where \mathcal{O} is an open neighborhood of 0 in \mathbb{R}^{d-1} .

The inverse diffeomorphism of Ψ is denoted by Φ , and we decompose it as

$$\Phi(x) = \begin{pmatrix} S(x) \\ Y(x) \end{pmatrix}, \quad (3.15)$$

where S takes its values in \mathbb{R}^{d-1} and Y takes its values in \mathbb{R} . If x belongs to

$$\mathcal{F} = \Psi(\overline{\mathcal{O}} \times [-r_1, r_1]),$$

the projection $P_K(x)$ is given by

$$P_K(x) = \Psi \begin{pmatrix} S(x) \\ Y(x)^+ \end{pmatrix}. \quad (3.16)$$

With these preparations, we are able to prove the main local estimates:

THEOREM 3.3. *For all $\bar{u} \in K$, for all $C_2 > 0$, there exist two positive numbers, $r_1 < r_2$ and three numbers $\tau > 0$, $h_1 > 0$ and $C_3 < \infty$ such that for all $h \in (0, h_1]$ and all $t_0 \in [0, T)$, for all U^0 and U^1 , satisfying the condition $E(\bar{u}, r_1, C_2, h)$, U^m is defined in $B(\bar{u}, r_2)$, for all $m \leq \lfloor \tau/h \rfloor$, and $|V^m|$ is bounded by C_3 independently of h for $0 \leq m \leq \lfloor \tau/h \rfloor - 1$.*

Proof. The theorem decomposes into an easy and a difficult part. The easy part is when \bar{u} belongs to the interior of K .

First case: $\bar{u} \in \text{int}(K)$. We choose $r_0 > 0$ as in the proof of lemma 3.2: the ball of center \bar{u} and radius $2r_0$ is included in K . The number C'_2 defined by (3.6) is equal to $4C_2$, and the numbers C_1 and L are given respectively by (3.7) and (3.8). We choose $r_1 = r_0/2$ and $r_2 = r_0$. Assume that τ satisfies the following inequalities:

$$\begin{aligned} 0 < \tau(C_1 + LC'_2) &< \min(C'_2 - C_2, r_0 - r_1), \\ \tau C_2 + \frac{\tau^2}{2}(C_1 + LC'_2) &\leq \frac{r_0}{2}. \end{aligned} \quad (3.17)$$

Then, if we write

$$n = \lfloor \tau/h \rfloor, \quad (3.18)$$

we shall prove by induction that for small enough h , there exists a unique solution of (1.16), for $0 \leq m \leq n$, which satisfies the estimate

$$|V^m| \leq C'_2.$$

We claim that for h small enough, we can find a solution of

$$U^{m+1} - 2U^m + U^{m-1} = h^2 F(t_m, U^{m-1}, U^m, (V^m + V^{m-1})/2, h) \quad (3.19)$$

which satisfies the estimates

$$\forall m \in \{0, \dots, n-1\}, \quad |U^m - \bar{u}| \leq r_1 + mh(C_1 + LC'_2), \quad (3.20)$$

$$\forall m \in \{0, \dots, n\}, \quad |V^m| \leq C_2 + mh(C_1 + LC'_2). \quad (3.21)$$

In this construction, we seek a solution without considering the constraints, and we prove eventually that they are satisfied.

It is clear that (3.20) holds for $m \leq 2$ and that (3.21) holds for $m \leq 1$. Assume that it holds up to some exponent $m < n$. Thanks to (3.17), we have the estimates

$$\begin{aligned} |U^{m-2} - \bar{u}| &\leq r_0, \quad |U^{m-1} - \bar{u}| \leq r_0, \quad |U^m - \bar{u}| \leq r_0, \\ |V^{m-2}| &\leq C'_2, \quad |V^{m-1}| \leq C'_2. \end{aligned}$$

Therefore, we may apply lemma 3.2 with $K = \mathbb{R}^d$: defining $C''_2 = 4C'_2$, we can find h_1 such that for $0 < h \leq h_1$, there exists a unique U^{m+1} such that

$$|U^{m+1} - U^m| \leq C''_2 h.$$

In particular, if L' is defined by (3.8), with C'_2 replaced by C''_2 , we infer from (3.19) that

$$|V^m| \leq |V^{m-1}| + h(C_1 + L'C''_2); \quad (3.22)$$

therefore, with the help of the induction assumption, we have the estimate:

$$|V^m| \leq C_2 + mh(C_1 + LC'_2) + h(L'C''_2 - LC'_2).$$

If h satisfies the inequality

$$C_2 + \tau(C_1 + LC'_2) + h(L'C''_2 - LC'_2) \leq C'_2,$$

we can see that in fact

$$|V^m| \leq C'_2,$$

and therefore, instead of (3.22), we have

$$|V^m| \leq C_2 + mh(C_1 + LC'_2).$$

Therefore, we have also

$$|U^{m+1} - \bar{u}| \leq \frac{r_0}{2} + (m+1)hC_2 + \frac{m(m+1)h^2}{2}(C_1 + hC'_2).$$

Thus, (3.20) and (3.21) hold. Let us prove that the vector W^m defined by (1.24) belongs to K : since

$$W^m - \bar{u} = U^{m-1} + \frac{2h}{1+e}V^{m-1} + \frac{h^2F^m}{1+e} - \bar{u}$$

we have the estimate

$$|W^m - \bar{u}| \leq r_0 + \frac{2hC'_2}{1+e} + \frac{h^2}{1+e}(C_1 + LC'_2);$$

thus, if $h \leq h_1$ and h_1 satisfies

$$\frac{2h_1C'_2}{1+e} + h_1^2(C_1 + LC'_2) \leq r_0,$$

W^m belongs to K , and the sequence U^m satisfies (1.16). This concludes the proof of the estimates in the first case. In particular, we can choose $C_3 = C'_2$.

Second case: $\bar{u} \in \partial K$. We define on \mathbb{R}^d a norm denoted by $\| \cdot \|$ as follows:

$$x = \begin{pmatrix} x' \\ x_d \end{pmatrix}, \quad \|x\| = \max(|x'|, |x_d|).$$

Pick $R_1 > 0$ such that Ψ is a diffeomorphism from an open neighborhood

$$\mathcal{B} = \{x : \|x - \Phi(\bar{u})\| \leq R_1\}$$

to its image and such that $\Psi(\mathcal{B})$ is included in an euclidean ball $B(\bar{u}, r_0)$ such that P_K is Lipschitz continuous on $B(\bar{u}, 2r_0)$; denote by γ the Lipschitz constant of P_K on $B(\bar{u}, 2r_0)$.

Define Λ by

$$\Lambda = \max \left\{ \sup \left\{ \frac{|D\Psi(x)x_1|}{\|x_1\|} : x \in \mathcal{B}, x_1 \neq 0 \right\}, \right. \\ \left. \sup \left\{ \frac{|D\Phi(u)u_1|}{\|u_1\|} : u \in \Psi(\mathcal{B}), u_1 \neq 0 \right\} \right\},$$

and

$$C_4 = \max \left\{ \sup \left\{ \frac{|D^2\Psi(x)x_1 \otimes x_2|}{2\|x_1\|\|x_2\|} : x \in \mathcal{B}, x_1 \neq 0, x_2 \neq 0 \right\}, \right. \\ \left. \sup \left\{ \frac{|D^2\Phi(u)u_1 \otimes u_2|}{2\|u_1\|\|u_2\|} : u \in \mathcal{B}, u_1 \neq 0, u_2 \neq 0 \right\} \right\}.$$

A continuity argument shows that the compact set $\Psi(\mathcal{B})$ contains the ball of radius R_1/Λ about \bar{u} .

We will give now a description of the scheme (1.16), (1.17) in the new coordinates $X^m = \Phi(U^m)$. Assume therefore that

$$U^{m+1}, U^m, U^{m-1}, W^m \text{ and } \frac{U^m + EU^{m-1}}{1+e} \text{ belong to } \mathcal{B}. \quad (3.23)$$

We know that (1.16) is equivalent to

$$\frac{U^{m+1} + eU^{m-1}}{1+e} = P_K(W^m); \quad (3.24)$$

We map (3.24) by Φ , and we calculate the Taylor expansion of either side of (3.24) around U^m . The left hand side of (3.24) can be rewritten as

$$U^m + h \frac{V^m - eV^{m-1}}{1+e}$$

and therefore

$$\begin{aligned} & \Phi \left(U^m + h \frac{V^m - eV^{m-1}}{1+e} \right) \\ &= \Phi(U^m) + D\Phi(U^m)h \frac{V^m - eV^{m-1}}{1+e} + I^m \end{aligned} \quad (3.25)$$

where

$$\|I^m\| \leq C_4 \left| \frac{h(V^m - eV^{m-1})}{1+e} \right|^2.$$

But

$$U^{m+1} = U^m + hV^m,$$

so that another Taylor expansion gives

$$\Phi(U^{m+1}) = \Phi(U^m) + D\Phi(U^m)hV^m + \widehat{I}^m$$

with

$$\|\widehat{I}^m\| \leq C_4 |hV^m|^2.$$

Thus

$$D\Phi(U^m)hV^m = \Phi(U^{m+1}) - \Phi(U^m) - \widehat{I}^m. \quad (3.26)$$

A similar calculation gives

$$-D\Phi(U^m)hV^{m-1} = \Phi(U^{m-1}) - \Phi(U^m) - \widetilde{I}^m, \quad (3.27)$$

with

$$\|\widetilde{I}^m\| \leq C_4 |hV^{m-1}|^2.$$

If we substitute (3.26) and (3.27) into (3.25), we find that

$$\Phi(U^{m+1}) = \frac{X^{m+1} + eX^{m-1}}{1+e} - \frac{\bar{I}^m}{1+e}$$

where

$$\bar{I}^m = \hat{I}^m + e\tilde{I}^m - (1+e)I^m,$$

and have the estimate

$$\|\bar{I}^m\| \leq C_4 h^2 (|V^m|^2 + e|V^{m-1}|^2 + (1+e)^{-1}|V^m - eV^{m-1}|^2). \quad (3.28)$$

Consider now the right hand side of (3.24). By definition of V^{m-1} , we have the identity

$$W^m = U^m + \frac{(1-e)hV^{m-1} + h^2F^m}{1+e}, \quad (3.29)$$

and a Taylor expansion gives

$$\Phi(W^m) = \Phi(U^m) + D\Phi(U^m) \frac{(1-e)hV^{m-1} + h^2F^m}{1+e} + J^m, \quad (3.30)$$

with

$$\|J^m\| \leq C_4 \left| \frac{(1-e)hV^{m-1} + h^2F^m}{1+e} \right|^2.$$

We substitute (3.27) into (3.30), and we obtain

$$\Phi(W^m) = \frac{2X^m - (1-e)X^{m-1} + h^2D\Phi(U^m)F^m}{1+e} + \frac{\bar{J}^m}{1+e},$$

where

$$\bar{J}^m = (1+e)J^m + (1-e)\tilde{I}^m,$$

so that

$$\|\bar{J}^m\| \leq C_4 \left[\frac{|(1-e)hV^{m-1} + h^2F^m|^2}{1+e} + (1-e)h^2|V^{m-1}|^2 \right].$$

We have to estimate $\|\bar{I}^m\| + \|\bar{J}^m\|$; by elementary inequalities,

$$\begin{aligned} \|I^m\| + \|J^m\| &\leq C_4 h^2 \left[\frac{2(1-e)^2|V^{m-1}|^2 + 2h^2|F^m|^2}{1+e} \right. \\ &\quad \left. + (1-e)|V^{m-1}|^2 + |V^m|^2 + e|V^{m-1}|^2 + \frac{2|V^m|^2 + 2e^2|V^{m-1}|^2}{1+e} \right]. \end{aligned}$$

The coefficient of $|V^{m-1}|^2$ in the above bracket is

$$\frac{2(1-e)^2}{1+e} + 1 + \frac{2e^2}{1+e},$$

and since for $e \in [0, 1]$, $(1 - e)^2 \leq 1 - e^2$, this coefficient is at most equal to 3. The coefficient of $|V^m|^2$ in the same bracket is at most equal to

$$1 + \frac{2}{1 + e},$$

which is also at most equal to 3. Therefore

$$\|\bar{I}^m\| + \|\bar{J}^m\| \leq C_4 h^2 [3|V^m|^2 + 3|V^{m-1}|^2 + 2h^2|F^m|^2]. \quad (3.31)$$

Thanks to the properties of P_K ,

$$\Phi\left(\frac{U^{m+1} + eU^{m-1}}{1 + e}\right) = \Phi(P_K W^m) = \begin{pmatrix} S(W^m) \\ Y(W^m)_+ \end{pmatrix}. \quad (3.32)$$

Define

$$s^m = S(U^m) = [X^m]', \quad y^m = Y(U^m) = X_d^m.$$

In these new coordinates, we have

$$s^{m+1} - 2s^m + s^{m-1} = h^2 \kappa^m, \quad (3.33)$$

$$y^{m+1} + ey^{m-1} - (2y^m - (1 - e)y^{m-1})^+ = h^2 \lambda^m, \quad (3.34)$$

and κ^m and λ^m are given by

$$\begin{aligned} h^2 \kappa^m &= [h^2 D\Phi(U^m)F^m + \bar{I}^m + \bar{J}^m]', \\ h^2 \lambda^m &= [2y^m - (1 - e)y^{m-1} + (h^2 D\Phi(U^m)F^m + \bar{J}^m)_d]^+ \\ &\quad - (2y^m - (1 - e)y^{m-1})^+ + \bar{I}_d^m. \end{aligned}$$

Therefore, we have the estimates:

$$\max(|\kappa^m|, |\lambda^m|) \leq \Lambda |F^m| + C_4 (3|V^m|^2 + 3|V^{m-1}|^2 + 2h^2|F^m|^2). \quad (3.35)$$

We define ξ^m and ζ^m by

$$\xi^m = \begin{pmatrix} \sigma^m \\ \eta^m \end{pmatrix} = \frac{X^{m+1} - X^m}{h}, \quad \zeta^m = \begin{pmatrix} \kappa^m \\ \lambda^m \end{pmatrix}.$$

Let now q be a number which satisfies

$$q > \Lambda C_2.$$

Let

$$C'_2 = \Lambda q(3\gamma + 1), \quad (3.36)$$

and let C_1 and L be respectively as in (3.7) and (3.8). If we assume beyond (3.23) that

$$\max(|V^{m-1}|, |V^m|) \leq C'_2, \quad (3.37)$$

we have the estimate

$$|F^m| \leq C_1 + L(|V^m| + |V^{m-1}|)/2;$$

by elementary inequalities,

$$|F^m|^2 \leq 2C_1^2 + L^2(|V^m|^2 + |V^{m-1}|^2),$$

and therefore, if we define

$$C_5 = \frac{\Lambda}{2} + C_1^2(\Lambda + 4h_1^2C_4), \quad C_6 = \left(\left(\frac{\Lambda}{2} + 2h_1^2C_4\right)L^2 + 3C_4\right)\Lambda^2,$$

we have shown that under assumptions (3.23) and (3.37), the following inequality holds:

$$\|\zeta^m\| \leq C_5 + C_6(\|\xi^m\|^2 + \|\xi^{m-1}\|^2). \quad (3.38)$$

Let τ be a number which satisfies the following inequalities:

$$\begin{aligned} \tau > 0, \quad \Lambda C_2 + (2\tau C_5 + 2C_6 q^2) < q, \\ 0 < \rho = \frac{R_1}{2} - \tau \Lambda C_2 - 2\tau^2(C_5 + 2C_6 q^2). \end{aligned} \quad (3.39)$$

Assume that initially

$$\max_{j=0,1,2} |U^j - \bar{u}| \leq R_1/(2\Lambda), \quad \max_{j=0,1} |U^{j+1} - U^j| \leq C_2 h. \quad (3.40)$$

We will prove by induction that if $n = \lfloor \tau/h \rfloor$, then for all $m \leq n$

$$\begin{aligned} \forall l \in \{0, \dots, m\}, \quad \|X^l - \Phi(\bar{u})\| &\leq \|X^0 - \Phi(\bar{u})\| \\ &\quad + lh\Lambda C_2 + 2l(l-1)h^2(C_5 + 2C_6 q^2), \\ \forall l \in \{0, \dots, m-1\}, \quad \|\xi^l\| &\leq \Lambda C_2 + 2lh(C_5 + 2C_6 q^2). \end{aligned} \quad (3.41)$$

For $m \leq 2$, assumptions (3.40) guarantee that (3.41) holds. The induction hypotheses imply that

$$\max_{j=m-2, m-1, m} \|X^j - \Phi(\bar{u})\| \leq R_1 - \rho, \quad \max_{j=m-2, m-1} \|X^{j+1} - X^j\| \leq qh,$$

and therefore, U^{m-2} , U^{m-1} and U^m belong to $\Psi(\mathcal{B})$. We may apply lemma (3.2) which guarantees the existence of U^{m+1} such that

$$|U^{m+1} - U^m| \leq (3\gamma + 1)\Lambda qh. \quad (3.42)$$

The ball of radius ρ/Λ about U^m is included in $\Psi(\mathcal{B})$; thus, if

$$(3\gamma + 1)\Lambda^2 qh \leq \rho,$$

U^{m+1} also belongs to $\Psi(\mathcal{B})$. Similarly,

$$\left| \frac{U^{m+1} + eU^{m-1}}{1+e} - U^m \right| \leq (3\gamma + 1)\Lambda qh,$$

and if

$$(3\gamma + 1\Lambda^2qh \leq \rho,$$

$(U^{m+1} + eU^{m-1})/(1 + e)$ belongs to $\Psi(\mathcal{B})$. Finally, thanks to the definition (3.36) of C'_2 , we have the inequality:

$$\left| \frac{U^{m+1} - U^{m-1}}{2h} \right| \leq \frac{(3\gamma + 2)\Lambda q}{2} \leq C'_2;$$

in virtue of the definitions (3.8) of L and (3.7) of C_1 , we have

$$|W^m - U^m| \leq \frac{1-e}{1+e}\Lambda qh + \frac{h^2}{1+e}(LC'_2 + C_1).$$

Once again, if

$$\frac{1-e}{1+e}\Lambda qh + \frac{h^2}{1+e}(LC'_2 + C_1) \leq \frac{\rho}{\Lambda},$$

W^m belongs to $\Psi(\mathcal{B})$. Thus (3.23) holds and we may apply the argument that followed.

By definition of σ^m and η^m , we have the inequalities

$$|\sigma^m| \leq |\sigma^{m-1}| + h \|\zeta^m\|,$$

and thanks to lemma 2.1

$$|\eta^m| \leq \max(|\eta^{m-1}|, e|\eta^{m-2}|) + h \|\zeta^m\| + h \|\zeta^{m-1}\|;$$

hence we infer that

$$\|\xi^m\| \leq \max(\|\xi^{m-1}\|, e\|\xi^{m-2}\|) + h \|\zeta^m\| + h \|\zeta^{m-1}\|, \quad (3.43)$$

and thanks to the induction hypothesis

$$\begin{aligned} \|\xi^m\| &\leq \Lambda C_2 + 2mh(C_5 + 2C_6q^2) - hC_6q^2 \\ &\quad + h(C_5 + C_6q^2) + hC_6\|\xi^m\|^2. \end{aligned} \quad (3.44)$$

The equation in a

$$hC_6a^2 - a + 2mh(C_5 + 2C_6q^2) - hC_6q^2 + h(C_5 + C_6q^2) + \Lambda C_2 = 0 \quad (3.45)$$

has two distinct real roots if

$$\Delta = 1 - 4hC_6(2mh(C_5 + 2C_6q^2) - hC_6q^2 + h(C_5 + C_6q^2) + \Lambda C_2)$$

is strictly positive; but this is always true if $0 < h \leq h_1$ and

$$1 > 4h_1C_6(2\tau(C_5 + 2C_6q^2) + \Lambda C_2).$$

The smallest of the two roots of (3.45) is inferior to q , since the substitution $a = q$ in (3.45) gives a negative left hand side; the largest of these two roots is at least equal to $1/(2hC_6)$; but relation (3.42) implies

$$\|\xi^m\| \leq \Lambda C'_2;$$

thus, if

$$h_1 \leq \frac{1}{2C_6\Lambda C_2'},$$

relation (3.44) implies $\|\xi^m\| \leq q$; if we substitute this inequality in the right hand side of (3.44), we find that the second inequality in (3.41) holds for $l = m$; the first inequality in (3.41) for $l = m + 1$ holds immediately, and the induction is proved. Thus, we can take as an upper bound of $|V^m|$ the number $C_3 = \Lambda q$; we can take also $r_1 = R_1/(2\Lambda)$ and $r_2 = \Lambda R_1$. \square

If we put together theorem 3.3 and lemma 3.1, we obtain an existence result:

THEOREM 3.4. *For all $(t_0, u_0, M(u_0)v_0) \in \mathbb{D}$, for all U^1 satisfying (1.14), there exists $\tau > 0$, $C_3 < \infty$ and h_1 such that for all $h \in (0, h_1]$, there exists a unique solution of (1.16) and (1.17) for all $m \leq \lfloor \tau/h \rfloor - 1$, which satisfies the estimate*

$$\forall l \leq n - 1, \quad |V^l| \leq C_3. \quad (3.46)$$

Proof. Let us check that U^0 and U^1 satisfy condition $E(u_0, r_1, C_2, h)$. Lemma 3.1 and assumption (1.14) on U^1 imply that

$$|U^1 - u_0|_{u_0} \leq h(|z(h)|_{u_0} + |v_0|_{u_0})$$

and

$$|U^2 - u_0|_{u_0} \leq h(2|v_0|_{u_0} + 1).$$

Choose $C_2 \geq (4|v_0|_{u_0} + 1)\|M(u_0)\|$; U^0 and U^1 satisfy condition $E(u_0, r_1, C_2, h)$ for small enough values of h . Then, it is clear that theorem 3.3 applies. \square

It is convenient to give a uniformized version of theorem 3.3:

THEOREM 3.5. *For all compact subset \mathcal{C} of K , for all $C_2 > 0$, there exist positive numbers r_1 , $r_2 > r_1$, τ , C_3 , and h_1 such that for all $t_0 \in [0, T)$, for all $\bar{u} \in \mathcal{C}$, for all $h \leq h_1$ and for all U^0 and U^1 satisfying condition $E(\bar{u}, r_1, C_2, h)$ relations (1.16) and (1.17) define uniquely under condition (3.46) the vectors U^m for $2 \leq m \leq \lfloor \min(\tau, T - t_0)/h \rfloor$.*

Proof. Any element u of \mathcal{C} is included in an open ball $\text{int}(B(u, r_1(u)))$ such that theorem 3.3 holds. We cover \mathcal{C} by a finite number of balls $\text{int}(B(u_j, r_1(u_j)/2))$ with associated numbers $r_2(u_j)$, $\tau(u_j)$, $h_1(u_j)$ and $C_3(u_j)$. If we let

$$r_1 = \frac{1}{2} \min\{r_1(u_j) : 1 \leq j \leq J\};$$

then any $\bar{u} \in \mathcal{C}$ belongs to a ball $B(u_j, r_1(u_j)/2)$, and in particular, $B(\bar{u}, r_1)$ is included in $B(u_j, r_1(u_j))$. If we take

$$\tau = \min_j \tau(u_j), \quad r_2 = \max_j r_2(u_j) \quad h_1 = \min_j h_1(u_j), \quad C_3 = \max_j C_3(u_j),$$

it is immediate that the theorem holds, thanks to theorem 3.3. \square

4. Estimates on the acceleration. In this section and the three following ones, we assume that there exist strictly positive numbers τ , C_3 and h_1 , and a subsequence

of times steps to which correspond solutions of the numerical scheme defined by (1.13), (1.14), (1.16) and (1.17), which satisfy the estimate, for all $h \leq h_1$:

$$\forall l \in \{0, P-1\}, \quad |U^{l+1} - U^l| \leq C_3 h \quad (4.1)$$

where

$$P = \lfloor \tau/h \rfloor$$

Here we estimate the discrete total variation of the sequence $(V^m)_m$. It is also convenient to define the function $w_h(t)$ on $[t_0, t_0 + \tau]$ by

$$w_h(t_m) = W^m, \quad w_h \text{ is continuous and it is affine} \\ \text{on each interval } [t_m, t_{m+1}), \text{ and constant on } [t_P, t_0 + \tau].$$

THEOREM 4.1. *Under assumption (4.1), there exists a constant C_7 such that for all $h \leq h_1$:*

$$\sum_{m=1}^{P-1} |V^m - V^{m-1}| \leq C_7. \quad (4.2)$$

Proof. Let \mathcal{C} be the compact set $K \cap B(u_0, C_3\tau)$ and let r_1 be as in theorem 3.5; cover \mathcal{C} with a finite number of balls $B(u_j, r_1/4)$; observe that, thanks to Ascoli–Arzelà’s theorem, the set \mathcal{W} of functions $(w_h)_{0 < h_1 \leq h}$ is relatively compact in $C^0([t_0, t_0 + \tau])$. The set of limit points of $(w_h)_{0 < h \leq h_1}$ as h tends to 0 is also a compact set, which we shall denote by \mathcal{W}_∞ . There exists a finite subset w^1, \dots, w^I of \mathcal{W}_∞ such that

$$\forall w \in \mathcal{W}_\infty : \inf\{\|w - w^i\|_{C^0[t_0, t_0 + \tau]} : 1 \leq i \leq I\} \leq r_1/4.$$

For each $i \in \{1, \dots, I\}$, it is possible to find a finite increasing sequence of times

$$0 = \tau(i, 0) < \dots < \tau(i, k) < \dots < \tau(i, \kappa(i)) = \tau$$

such that

$$w^i([\tau(i, k), \tau(i, k+1)]) \subset B(u_{j(i, k)}, r_1/4).$$

Thus, for all $w \in \mathcal{W}_\infty$,

$$w([\tau(i, k), \tau(i, k+1)]) \subset B(u_{j(i, k)}, r_1/2).$$

Therefore, we can decrease h_1 so that

$$\forall h \in (0, h_1], \quad \exists i \in \{1, \dots, I\}, \quad \forall k \in \{1, \dots, \kappa(i)\}, \\ \forall t \in [\tau(i, k), \tau(i, k+1)] \quad w_h(t) \in B(u_{j(i, k)}, 3r_1/4),$$

and thanks to (3.29) and to (4.1), we can decrease h_1 such that

$$\forall h \in (0, h_1], \quad \exists i \in \{1, \dots, I\}, \quad \forall k \in \{1, \dots, \kappa(i) - 1\}, \\ \forall l \in \{\lfloor \tau(i, k)/h \rfloor, \dots, \lfloor \tau(i, k+1)/h \rfloor\}, \quad U^l \in B(u_{j(i, k)}, r_1).$$

We simplify the notations by letting

$$P = \lfloor \tau(i, k)/h \rfloor, \quad Q = \lfloor \tau(i, k+1)/h \rfloor,$$

and we take C_1 be as in (3.7), where \bar{u} is set equal to $u_{j(i, k)}$, r_0 is set equal to r_1 and C'_2 is set equal to C_3 .

Now, we have to consider two cases:

First case: $B(\bar{u}, r_1) \cap \partial K = \emptyset$. We have the inequality

$$|F^m| \leq C_1 + LC_3, \quad (4.3)$$

hence, thanks to (1.16), we have the inequality

$$|V^m - V^{m-1}| \leq h(C_1 + LC_3),$$

and therefore

$$\sum_{m=P+1}^Q |V^m - V^{m-1}| \leq (\tau(i, k+1) + 2h - \tau(i, k))(C_1 + LC_3). \quad (4.4)$$

Case 2: $B(\bar{u}, r_1) \cap \partial K \neq \emptyset$. We observe that thanks to (3.38), we have the estimate

$$\forall m \in \{P+1, \dots, Q-1\}, \quad \max(|\kappa^m|, |\lambda^m|) \leq C_9, \quad (4.5)$$

where

$$C_9 = C_5 + 2C_6\Lambda^2C_3^2.$$

The estimates on the first $d-1$ components of the velocity in the straightened coordinates are immediate:

$$\sum_{m=P+1}^Q \left| \frac{s^{m+1} - s^m}{h} - \frac{s^m - s^{m-1}}{h} \right| \leq (\tau(i, k+1) + 2h - \tau(i, k))C_9. \quad (4.6)$$

In order to estimate the last coordinate, we partition $\{P+1, \dots, Q\}$ as follows:

$$\begin{aligned} \mathcal{P} &= \{m \in \{P+1, \dots, Q\} : 2y^m - (1-e)y^{m-1} < 0\}, \\ \mathcal{P}' &= \{P+1, \dots, Q\} \setminus \mathcal{P}. \end{aligned}$$

We write \mathcal{P} as an union of discrete intervals:

$$\mathcal{P} = \bigcup_{l=1}^{\ell} \{p(l), \dots, q(l)\}, \quad p(l) - 1 \notin \mathcal{P}, \quad q(l) + 1 \notin \mathcal{P}.$$

If η^m is defined as in (2.2), we observe that for $m \in \mathcal{P}'$,

$$|\eta^m - \eta^{m-1}| \leq C_9 h,$$

so that

$$\sum_{m \in \mathcal{P}'} |\eta^m - \eta^{m-1}| \leq hC_9 |\mathcal{P}'|.$$

If m belongs to \mathcal{P} , we observe that

$$\eta^m - \eta^{m-1} = h\lambda^m + (2y^m - (1-e)y^{m-1})^- \quad (4.7)$$

and therefore, by the triangle inequality,

$$|\eta^m - \eta^{m-1}| \leq hC_9 + (2y^m - (1-e)y^{m-1})^-,$$

and using (4.7) again,

$$|\eta^m - \eta^{m-1}| \leq 2hC_9 + \eta^m - \eta^{m-1}. \quad (4.8)$$

We observe that we have the elements of a telescoping sum: we sum (4.8) for m varying from $p(l) + 1$ to $q(l)$, and we obtain

$$\sum_{m=p(l)}^{q(l)} |\eta^m - \eta^{m-1}| \leq 2hC_9(q(l) - p(l)) + \eta^{q(l)} - \eta^{p(l)}. \quad (4.9)$$

Now, we sum (4.9) from $l = 1$ to ℓ , which yields

$$\begin{aligned} \sum_{l=1}^{\ell} \sum_{m=p(l)}^{q(l)} |\eta^m - \eta^{m-1}| \\ \leq 2C_9(Q - P) - \eta^{p(1)} + \eta^{q(\ell)} - \sum_{l=2}^{\ell} \eta^{p(l)} - \eta^{q(l-1)}. \end{aligned}$$

But the terms $\eta^{p(l)} - \eta^{q(l-1)}$ can be estimated, since they correspond to a summation over \mathcal{P}' :

$$|\eta^{p(l)} - \eta^{q(l-1)}| \leq C_9 h(p(l) - q(l)).$$

Therefore, we have proved that

$$\sum_{m=P+1}^Q |\eta^m - \eta^{m-1}| \leq 3C_9 h(Q - P) + 2C_3 \Lambda.$$

Summarizing this relation with (4.6), we can see that

$$\sum_{m=P+1}^Q |V^m - V^{m-1}| \leq \Lambda C_9(4\tau(i, k+1) - 4\tau(i, k) + 8h) + 2C_3 \Lambda. \quad (4.10)$$

Relations (4.4) and (4.10) do not depend on $h \leq h_1$; since we have only a finite number of these estimates, the theorem is proved. \square

5. Variational properties of the limit of the numerical scheme. In this section, we work under the assumption (4.1). Recall that $n = \lfloor \tau/h \rfloor$. We define a function u_h by affine interpolation, as follows:

$$\begin{cases} u_h(t) = U^m + (t - mh) \frac{U^{m+1} - U^m}{h} \\ \quad \text{for } t \in [mh, (m+1)h), 0 \leq m \leq n-1, \\ u_h(t) = U^n \quad \text{for } t \in [nh, \tau]. \end{cases} \quad (5.1)$$

We also define a measure F_h as the following sum of Dirac masses:

$$F_h(t) = \sum_{m=1}^n h F^m \delta(t - mh). \quad (5.2)$$

In this section we prove that the sequence $(u_h)_h$ converges in an appropriate sense to a function u which satisfies (1.3) to (1.5b) with τ instead of T . We delay the proof of (1.6), the transmission condition at impacts, to a later section.

There are three steps in the convergence proof: the first is to prove that the limit u exists in an appropriate sense and takes its values in K ; in the second step, we show that \dot{u}_h is of bounded variation uniformly in h and that F_h converges to $M(u)^{-1}f(\cdot, u, M(u)\dot{u})$ weakly in the space of \mathbb{R}^d -valued measures. The last step is the characterization of the measure $\mu = M(u)\ddot{u} - f(\cdot, u, M(u)\dot{u})$: there we show that μ satisfies conditions (1.4a), (1.4b) and (1.4c).

LEMMA 5.1. *From all sequence of functions $(u_h)_h$ indexed by a sequence h tending to 0, it is possible to extract a subsequence, still denoted by $(u_h)_h$ such that*

$$u_h \rightarrow u \quad \text{in } C^0([t_0, t_0 + \tau]) \text{ strong,} \quad (5.3)$$

$$\dot{u}_h \rightarrow \dot{u} \quad \text{in } L^\infty([t_0, t_0 + \tau]) \text{ weak}^*. \quad (5.4)$$

The function u takes its values in K .

Proof. Thanks to assumption (4.1), we know that $(u_h)_{0 < h \leq h_1}$ is uniformly Lipschitz continuous over $[t_0, t_0 + \tau]$. Therefore, we may extract a subsequence, still denoted by u_h , such that (5.3) and (5.4) hold. Thus u belongs to $W^{1,\infty}([t_0, t_0 + \tau]) \cap C^0([t_0, t_0 + \tau])$, which means that u is a Lipschitz continuous function [3]. For all m belonging to $\{1, \dots, n\}$, we have:

$$Z^m = \frac{U^{m+1} + eU^{m-1}}{1 + e} = U^m + h \frac{V^m - eV^{m-1}}{1 + e}, \quad (5.5)$$

hence $U^m = Z^m - h(V^m - eV^{m-1})/(1 + e)$. By definition of the scheme, we have $Z^m = P_K(W^m)((1.26))$, and thus Z^m belongs to K . It follows that, for all $m \in \{1, \dots, n\}$, the euclidean distance between U^m and K can be estimated as follows:

$$\min\{|U^m - u| : u \in K\} \leq h |V^m - eV^{m-1}| / (1 + e) \leq hC_3. \quad (5.6)$$

Thanks to the definition (5.1), we can see that for all $t \in [t_0, t_0 + \tau]$ the euclidean distance between $u_h(t)$ and K is estimated by $2hC_3$. This allows us to pass to the limit when h tends to 0 and to conclude. \square

Next lemma describes the convergence of the measures involved in our problem; we denote by $M^1((t_0, t_0 + \tau))$ the space of bounded measures over $(t_0, t_0 + \tau)$ with values in \mathbb{R}^d .

LEMMA 5.2. *The measures \ddot{u}_h and F_h converge weakly in $M^1((t_0, t_0 + \tau))$ respectively to \ddot{u} and $M(u)^{-1}f(\cdot, u, M(u)\dot{u})$.*

Proof. The measure \ddot{u}_h is a sum of Dirac measures on $(t_0, t_0 + \tau)$, more precisely, we have:

$$\ddot{u}_h(t) = \sum_{m=1}^n (V^m - V^{m-1})\delta(t - mh) - V^n\delta(t - (n+1)h), \quad (5.7)$$

and the total variation of \dot{u}_h on $(t_0, t_0 + \tau)$ is estimated by

$$TV(\dot{u}_h) \leq \sum_{m=1}^n |V^m - V^{m-1}| + |V^n|. \quad (5.8)$$

Theorem 4.1 implies that $(\dot{u}_h)_{0 < h \leq h_1}$ is a bounded family in $BV((t_0, t_0 + \tau))$, the space of functions of bounded variation over $(t_0, t_0 + \tau)$, with values in \mathbb{R}^d . Using

Helly's theorem, we can extract another subsequence $(\dot{u}_h)_h$ which converges, except perhaps on a countable set of points, to a function of bounded variation. Hence

$$\dot{u} \in BV((t_0, t_0 + \tau)).$$

Moreover,

$$\ddot{u}_h \rightarrow \ddot{u} \quad \text{weakly in } M^1((t_0, t_0 + \tau)).$$

Lebesgue's theorem implies that \dot{u}_h converges to \dot{u} in $L^1(t_0, t_0 + \tau)$. We extend \dot{u}_h and \dot{u} to \mathbb{R} by 0 outside of $(t_0, t_0 + \tau)$ and still denote the respective extensions by \dot{u}_h and \dot{u} . The set $\{\dot{u}_h : h \in (0, h_1]\} \cup \{\dot{u}\}$ is a compact subset of $L^1(\mathbb{R})$. The classical characterization of compact subsets of $L^1(\mathbb{R})$ [6] implies that

$$\lim_{\theta \rightarrow 0} \sup_{0 < h \leq h_1} \int_{\mathbb{R}} |\dot{u}_h(t - \theta) - \dot{u}_h(t)| \, dt = 0. \quad (5.9)$$

Letting $\theta = h$, we can see that $\dot{u}_h(\cdot - h)$ converges to \dot{u} in $L^1(\mathbb{R})$. Let us define an approximate velocity v_h on \mathbb{R} by

$$v_h(t) = \frac{\dot{u}_h(t - h + 0) + \dot{u}_h(t + 0)}{2}. \quad (5.10)$$

The sequence v_h converges to \dot{u} in $L^1(\mathbb{R})$. Moreover, for all $t \in [t_m, t_{m+1})$ and for all $m \in \{1, \dots, n\}$, we have the identity

$$v_h(t) = \frac{V^m + V^{m-1}}{2}. \quad (5.11)$$

We have immediately the following estimates for all $t \in (t_0, t_0 + \tau)$ and all $h \in (0, h_1]$:

$$|v_h(t)| \leq C_3; \quad |u_h(t) - u_0| \leq C_3(t - t_0) \leq C_3\tau. \quad (5.12)$$

Let ψ be a continuous function over $[0, T]$ with compact support included in $(t_0, t_0 + \tau)$. For all small enough h , the support of ψ is included in $[t_0 + h, t_0 + nh]$. The duality product $\langle F_h, \psi \rangle$ has the expression

$$\langle F_h, \psi \rangle = \sum_{m=1}^n h \psi(t_0 + mh)^T F^m. \quad (5.13)$$

We wish to compare the expression (5.13) to

$$\int_{t_0}^{t_0 + \tau} \psi^T M(u)^{-1} f(\cdot, u, M(u)\dot{u}) \, dt. \quad (5.14)$$

We compare the right hand side of (5.13) which is basically a numerical quadrature by the formula of rectangles to an appropriate integral. Let us rewrite the individual terms of the right hand side of (5.13) as

$$h \psi(t_m)^T F^m = \int_{t_m}^{t_{m+1}} \psi(t)^T F^m \, dt + \int_{t_m}^{t_{m+1}} (\psi(t_m) - \psi(t))^T F^m \, dt. \quad (5.15)$$

Consider the second term on the right hand side of (5.15): we have already proved (see relations (4.3) and (4.5)) that there exists a constant C_8 independent of m and $h \leq h_1$ such that

$$\max_{0 \leq m \leq n} |F^m| \leq C_8. \quad (5.16)$$

Denoting by ω_ψ the modulus of continuity of ψ we can see that

$$\left| \int_{t_m}^{t_{m+1}} (\psi(t_m) - \psi(t))^T F^m dt \right| \leq C_8 \omega_\psi(h) h. \quad (5.17)$$

We consider now the first term on the right hand side of (5.15), which we would like to compare to expression (5.14). Thanks to the consistence assumption (1.12) have the following inequalities, for all $t \in [t_m, t_{m+1})$, and all $n \in \{1, \dots, n\}$:

$$\begin{aligned} & |F^m - M(u_h(t))^{-1} f(t, u_h(t), M(u_h(t))v_h(t))| \\ & \leq |F(t_m, U^m, U^{m-1}, v_h(t_m), h) - F(t_m, u_h(t), u_h(t), v_h(t_m), h)| \\ & + |F(t_m, u_h(t), u_h(t), v_h(t_m), h) - F(t_m, u_h(t), u_h(t), v_h(t), 0)| \\ & + |M(u_h(t))^{-1} [f(t_m, u_h(t), M(u_h(t))v_h(t)) - \\ & f(t, u_h(t), M(u_h(t))v_h(t))]|. \end{aligned}$$

For all $t \in [t_0, t_0 + \tau]$, let us define

$$p_h(t) = M(u_h(t))v_h(t).$$

Denote by \mathcal{D} the set

$$\begin{aligned} \mathcal{D} = \{ & (t, u_1, u_2, v, h) : 0 \leq t \leq T, \quad |u_1 - u_0| \leq C_3 \tau, \\ & |u_2 - u_0| \leq C_3 \tau, \quad |v| \leq C_3, \quad 0 \leq h \leq h_1 \}. \end{aligned}$$

Let L be the Lipschitz constant of $(u_1, u_2) \mapsto F(t, u_1, u_2, v, h)$ restricted to \mathcal{D} and let ω_F be the modulus of continuity of F on \mathcal{D} . With these notations, we can see that

$$\begin{aligned} & |F^m - M(u_h(t))^{-1} f(t, u_h(t), M(u_h(t))\dot{u}_h(t))| \\ & \leq L(|U^m - u_h(t)| + |U^{m-1} - u_h(t)|) + 2\omega_F(h). \end{aligned} \quad (5.18)$$

Since M is of class C^3 in \mathbb{R}^d , $(u_h)_h$ converges strongly in $C^0([t_0, t_0 + \tau])$ and $(v_h)_h$ converges strongly to \dot{u} in $L^1(\mathbb{R})$ and almost everywhere on $(t_0, t_0 + \tau)$, the sequence $(p_h)_h$ also converges strongly in $L^1(\mathbb{R})$ and almost everywhere on $(t_0, t_0 + \tau)$ to $p = M(u)\dot{u}$. We see that $M(u_h)^{-1} f(\cdot, u_h, p_h)$ tends to $M(u)^{-1} f(\cdot, u, p)$ strongly in $L^1(t_0, t_0 + \tau)$ and almost everywhere on $(t_0, t_0 + \tau)$. We summarize relations (5.17) and (5.18) together with the above convergence result, and we find that

$$\begin{aligned} & \left| \langle F_h, \psi \rangle - \int_{t_0}^{t_0 + \tau} \psi^T M(u)^{-1} f(\cdot, u, M(u)\dot{u}) dt \right| \\ & \leq \int_{t_0}^{t_0 + \tau} |M(u_h)^{-1} f(\cdot, u_h, M(u_h)v_h) - M(u)^{-1} f(\cdot, u, M(u)\dot{u})| |\psi| dt \\ & + C_8 \omega_\psi(h) \tau + (3LC_2 h + 2\omega_F(h)) \int_0^T |\psi| dt, \end{aligned}$$

which concludes the proof. \square

Let us prove now that the measure μ has the required variational properties:

LEMMA 5.3. *The measure μ satisfies properties (1.4a), (1.4b) and (1.4c).*

Proof. Define

$$\mu_h = M(u_h)(\ddot{u}_h - F_h);$$

μ_h is a sum of Dirac measures on $(t_0, t_0 + \tau)$; more precisely

$$\begin{aligned} \mu_h &= \sum_{m=1}^n M(U^m)(V^m - V^{m-1} - hF^m)\delta(t - mh) \\ &\quad - M(U^{n+1})V^n\delta(t - (n+1)h). \end{aligned}$$

With all the previous results, we know that μ_h converges to $\mu = M(u)\ddot{u} - f(\cdot, u, p)$ weakly in $M^1((t_0, t_0 + \tau))$. Let us prove property (1.4a). Assume that τ_0 is a point of $(t_0, t_0 + \tau)$ such that $u(\tau_0)$ belongs to the interior of K . Then, by continuity of u , there exist $\varepsilon > 0$ and $\rho > 0$ such that

$$\inf\{|u(t) - x| : |t - \tau_0| \leq \varepsilon, x \in \partial K\} \geq 3\rho.$$

Since the sequence $(u_h)_h$ converges uniformly to u as h tends to 0, we can decrease h_1 so that

$$\inf\{|u_h(t) - x| : |t - \tau_0| \leq \varepsilon, 0 < h \leq h_1, x \in \partial K\} \geq 2\rho.$$

Relation (1.24) implies the identity

$$W^m = U^m + \frac{1-e}{1+e}hV^{m-1} + \frac{h^2}{1+e}F^m. \quad (5.19)$$

Relations (5.19) and (5.16) imply that

$$|W^m - U^m| \leq hC_3 \frac{1-e}{1+e} + \frac{h^2C_8}{1+e}. \quad (5.20)$$

Possibly decreasing h_1 , we have thus

$$\inf\{|W^m - x| : |t_m - \tau_0| \leq \varepsilon, 0 < h \leq h_1, x \in \partial K\} \geq \rho.$$

This proves that the support of μ_h does not intersect the open set $(\tau_0 - \varepsilon, \tau_0 + \varepsilon)$, and therefore, relation (1.4a) holds. Assume now that $u_1 = u(t_1)$ belongs to ∂K , and let $B(u_1, R_1)$ be a ball having the properties of theorem 3.3; assume that the image of (τ_1, τ_2) by u_h and w_h is included in this ball for all small enough h . We rewrite conditions (1.4b) and (1.4c) as follows: for all continuous function ψ with compact support included in $(t_0, t_0 + \tau)$ and taking its values in \mathbb{R}^d the following implication holds:

$$\forall t \in (t_0, t_0 + \tau), \quad d\phi(u(t))\psi(t) \geq 0 \implies \langle \mu, \psi \rangle \geq 0. \quad (5.21)$$

In particular, if $d\phi(u(t))\psi(t)$ vanishes for all $t \in (t_0, t_0 + \tau)$, then $\langle \mu, \psi \rangle$ also vanishes.

The reader will check the equivalence of (1.4b) and (1.4c) with (5.21). We infer from relation (5.6) that

$$|Y(U^m)| \leq \Lambda h C_3;$$

the above relation together with (5.20) imply that there exists a constant C_{10} such that

$$|Y(W^m)| \leq hC_{10}.$$

Since (5.21) is local, it is enough to check it in the neighborhood of any $t_1 \in (t_0, t_0 + \tau)$. Let

$$P = \lceil \tau_1/h \rceil, \quad Q = \lfloor \tau_2/h \rfloor,$$

and

$$\mathcal{P} = \{m \in \{P, \dots, Q\} : W^m \notin K\}, \quad \mathcal{P}' = \{P, \dots, Q\} \setminus \mathcal{P}.$$

We observe that if m belongs to \mathcal{P}' , then

$$V^m - V^{m-1} - hF^m = 0.$$

Therefore, we have the identity:

$$\begin{aligned} & \sum_{m=P}^Q \langle V^m - V^{m-1} - hF^m, \psi(t_m) \rangle_{U^m} \\ &= \sum_{m \in \mathcal{P}} \langle V^m - V^{m-1} - hF^m, \psi(t_m) \rangle_{U^m}. \end{aligned}$$

We recall relation (1.29). Relation (3.32) implies that

$$\Phi(Z^m) - \Phi(W^m) = \begin{pmatrix} 0 \\ Y(W^m)^- \end{pmatrix},$$

and therefore

$$\left| Z^m - W^m - D\Psi(W^m) \begin{pmatrix} 0 \\ Y(W^m)^- \end{pmatrix} \right| \leq C_4 |Z^m - W^m|^2.$$

On the other hand, the definition of Ψ is such that the d -th column of $D\Psi(Z^m)$ is equal to $N(Z^m)$; therefore

$$\begin{aligned} & \left| D\Psi(W^m) \begin{pmatrix} 0 \\ Y(W^m)^- \end{pmatrix} - N(Z^m)Y(W^m)^- \right| \\ & \leq 2C_4 |Z^m - W^m| Y(W^m)^-. \end{aligned}$$

We infer from the above estimates that

$$\begin{aligned} & |Z^m - W^m - Y(W^m)^- N(Z^m)| \\ & \leq C_4 (2Y(W^m)^- + |Z^m - W^m|) |Z^m - W^m| \\ & \leq \frac{C_4 h}{1+e} |V^m - V^{m-1} - hF^m| (2 + \Lambda) hC_{10}, \end{aligned}$$

and thus, there exists C_{11} such that for all $m \in \mathcal{P}$:

$$|Z^m - W^m - Y(W^m)^- N(Z^m)| \leq h^2 C_{11} |V^m - V^{m-1} - hF^m|.$$

We can see now that

$$\begin{aligned}
& \sum_{m \in \mathcal{P}} \langle V^m - V^{m-1} - hF^m, \psi(t_m) \rangle_{U^m} \\
&= \frac{1+e}{h} \sum_{m \in \mathcal{P}} \langle Z^m - W^m, \psi(t_m) \rangle_{U^m} \\
&\geq \frac{1+e}{h} \sum_{m \in \mathcal{P}} Y(W^m)^- \langle N(Z^m), \psi(t_m) \rangle_{U^m} \\
&\quad - C_{11}h(1+e) \max_{P \leq m \leq Q} (\|M(U^m)\| |\psi(t_m)| \sum_{m \in \mathcal{P}} |V^m - V^{m-1} - hF^m|),
\end{aligned}$$

which implies by a straightforward passage to the limit that $\langle u, \psi \rangle$ is non negative. This concludes the proof of the lemma. \square

6. Transmission of energy during impact. The basic assumption is still the one made at the beginning of Section 4.

Let $\bar{\tau} \in (0, \tau)$ be such that $u(\bar{\tau})$ belongs to ∂K . Write $\bar{t} = t_0 + \bar{\tau}$. We decompose $p(\bar{t} \pm 0)$ into a normal component $p_N(\bar{t} \pm 0)$ belonging to $\mathbb{R}d\phi(u(\bar{t}))$ and a tangential part $p_T(\bar{t} \pm 0)$ belonging to the orthogonal of $d\phi(u(\bar{t}))$ in the cotangent metric at $u(\bar{t})$.

In this section, we shall prove that

$$p_T(\bar{t} + 0) = p_T(\bar{t} - 0) \text{ and } p_N(\bar{t} + 0) = -ep_N(\bar{t} - 0), \quad (6.1)$$

where e is the restitution coefficient of the problem.

The conservation of the tangential component of the impulsion is proved in next lemma:

LEMMA 6.1. *Assume that $\bar{\tau} \in (0, \tau)$ is such that $u(\bar{\tau})$ belongs to ∂K . Then*

$$p_T(\bar{t} + 0) = p_T(\bar{t} - 0).$$

Proof. Thanks to lemma 5.3, we know that

$$M(u)\ddot{u} = \mu + f(\cdot, u, p), \quad (6.2)$$

and that there exists a nonnegative measure λ such that

$$\mu = \lambda d\phi(u). \quad (6.3)$$

We take the measure of the set $\{\bar{t}\}$ by the two sides of (6.2), and we find that

$$M(u(\bar{t}))(\dot{u}(\bar{t} + 0) - \dot{u}(\bar{t} - 0)) = \mu(\{\bar{t}\}),$$

which implies immediately that $p(\bar{t} + 0) - p(\bar{t} - 0)$ is parallel to $d\phi(u(\bar{t}))$ and proves the lemma. \square

Let $\bar{u} = u(\bar{t})$ and let $B(\bar{u}, r_1)$ and $B(\bar{u}, r_1)$ have the properties of theorem 3.3. There exists an interval $[\tau_{-5}, \tau_2]$ containing $\bar{\tau}$ in its interior such the for all small enough h , $u_h([t_0 + \tau_{-5}, t_0 + \tau_2])$ is included in $B(u_1, r_1)$.

The apparently strange notations τ_{-5} and τ_2 have been chosen in view of the upcoming construction of lemmas 6.3 and 6.4, where we will consider relative times

$$\tau_{-5} < \dots < \tau_{-1} < \bar{\tau} < \tau_1 < \tau_2.$$

Define

$$P = \lceil \tau_{-5}/h \rceil + 1, \quad Q = \lfloor \tau_2/h \rfloor - 1,$$

and let x_h be obtained from the X^m by affine interpolation, for $P \leq m \leq Q$. We infer from estimates (4.1) and (4.2) the estimates

$$\begin{aligned} \max_{P \leq m \leq Q} \left| \frac{X^{m+1} - X^m}{h} \right| &\leq \Lambda C_3, \\ \sum_{m=P}^Q \left| \frac{X^{m+1} - X^m}{h} - \frac{X^m - X^{m-1}}{h} \right| &\leq \Lambda C_7. \end{aligned}$$

Therefore, we have the following convergences

$$\begin{aligned} x_h &\rightarrow x \text{ strongly in } C^0([t_0 + \tau_{-5}, t_0 + \tau_2]); \\ \dot{x}_h &\rightarrow \dot{x} \text{ except on a countable set and weakly } * \\ &\quad \text{in } L^\infty([t_0 + \tau_{-5}, t_0 + \tau_2]); \\ \ddot{x}_h &\rightarrow \ddot{x} \text{ weakly in } M^1([t_0 + \tau_{-5}, t_0 + \tau_2]). \end{aligned}$$

Write for all $h \leq h_1$

$$x_h = \begin{pmatrix} s_h \\ y_h \end{pmatrix}, \quad x = \begin{pmatrix} s \\ y \end{pmatrix},$$

where the s_h 's and s take their values in \mathbb{R}^{d-1} and the y_h 's and y are real valued functions. We do not have $x_h = \Phi(u_h)$, because x_h is a linear interpolation of the sequence $X^m = \Phi(U^m)$, and $\Phi(u_h)$ is the image of the linear interpolation of the sequence U^m . However, we can estimate the difference $x_h - \Phi(u_h)$.

LEMMA 6.2. *For all $t \in [t_0 + \tau_{-5}, t_0 + \tau_2]$, belonging to $[t_m, t_{m+1}]$, we have:*

$$x_h(t) - \Phi(u_h(t)) \leq 2C_4 C_3^2 h \min(t - t_m, t_{m+1} - t).$$

Proof. We observe that

$$x_h(t_m) = X^m,$$

and that

$$\begin{aligned} &\left| \frac{d}{dt} [x_h(t) - \Phi(u_h(t))] \right|_{t=t_m+0} \\ &= \left| \frac{\Phi(U^{m+1}) - \Phi(U^m) - h D\Phi(U^m)V^m}{h} \right| \\ &\leq h C_3^2 C_4. \end{aligned}$$

Moreover, for all $t \in [t_m, t_{m+1})$

$$\begin{aligned} &\left| \frac{d^2}{dt^2} [x_h(t) - \Phi(u_h(t))] \right| \\ &= |D^2\Phi(U^m + (t - t_m)V^m)V^m \otimes V^m| \leq 2C_3^2 C_4. \end{aligned}$$

Therefore, a straightforward integration yields

$$|x_h(t) - \Phi(u_h(t))| \leq C_3^2 C_4 (h(t - t_m) + (t - t_m)^2),$$

which implies

$$|x_h(t) - \Phi(u_h(t))| \leq 2C_3^2 C_4 h(t - t_m).$$

We can write the analogous estimate on the interval $[t, t_{m+1})$, which concludes the proof. \square

As a consequence of lemma 6.2 we obtain:

$$\forall t \in [t_0 + \tau_{-5}, t_0 + \tau_2], \quad x(t) = \Phi(u(t)),$$

and

$$\forall t \in (t_0 + \tau_{-5}, t_0 + \tau_2), \quad \dot{x}(t \pm 0) = D\Phi(u(t))\dot{u}(t \pm 0).$$

In virtue of relation (3.14),

$$\dot{u}(\bar{t} \pm 0) = \begin{pmatrix} \dot{s}(\bar{t} \pm 0) \\ 0 \end{pmatrix} + \dot{y}(\bar{t} \pm 0)N(0).$$

We can rewrite this relation in terms of p_N and p_T :

$$p_T(\bar{t} \pm 0) = M(0) \begin{pmatrix} \dot{s}(\bar{t} \pm 0) \\ 0 \end{pmatrix}, \quad p_N(\bar{t} \pm 0) = \dot{y}(\bar{t} \pm 0)M(0)N(0)$$

Lemma 6.1 implies $\dot{s}(\bar{t} + 0) = \dot{s}(\bar{t} - 0)$. In order to achieve the proof of relation (6.1), we will prove the scalar relation

$$\dot{y}(\bar{t} + 0) = -e\dot{y}(\bar{t} - 0). \quad (6.4)$$

We will do this by performing a precise analysis of the transmission of energy on the scheme (2.1). The measure \dot{y}_h is a sum of Dirac measures on $(t_0 + \tau_{-5}, t_0 + \tau_2)$. We define two measures ω_h and λ_h on $(t_0 + \tau_{-5}, t_0 + \tau_2)$ by

$$\omega_h = \sum_{m=P}^Q \frac{(-2y^m + (1-e)y^{m-1})^+}{h} \delta(t - mh),$$

and

$$\lambda_h(t) = \sum_{m=P}^Q h\lambda^m \delta(t - mh).$$

We have

$$\ddot{y}_h = \omega_h + \lambda_h,$$

and it is obvious that ω_h is a non-negative measure.

Since the real numbers λ^m are bounded independently of h and n , the measure by $|\lambda_h|$ of any subinterval $[a, b]$ of $(t_0 + \tau_{-5}, t_0 + \tau_2)$ is bounded by $C(b - a + h)$,

and it is clear therefore that there exists a function $\lambda \in L^\infty(t_0 + \tau_{-5}, t_0 + \tau_2)$ and a subsequence λ_h converging to λ in the weak topology of $M^1((t_0 + \tau_{-5}, t_0 + \tau_2))$.

The measure ω_h converges in the weak topology of $M^1((t_0 + \tau_{-5}, t_0 + \tau_2))$ to a non-negative measure ω , and in the limit

$$\ddot{y} = \omega + \lambda, \quad (6.5)$$

while

$$|\lambda|_{L^\infty} \leq C_9. \quad (6.6)$$

Since y is non-negative on $(t_0 + \tau_{-5}, t_0 + \tau_2)$ and $y(\tau_0)$ vanishes, we must have

$$\dot{y}(\bar{t} + 0) \geq 0, \quad \dot{y}(\bar{t} - 0) \leq 0.$$

On the other hand, $\dot{y}(\bar{t} + 0) - \dot{y}(\bar{t} - 0)$ is equal to $\omega(\{\bar{t}\})$; if $\omega(\{\bar{t}\})$ vanishes, we have

$$\dot{y}(\bar{t} + 0) = \dot{y}(\bar{t} - 0) = 0,$$

and the identity

$$\dot{y}(\bar{t} + 0) = -e\dot{y}(\bar{t} - 0)$$

holds. Therefore, the only interesting case is when

$$\omega(\{\bar{t}\}) > 0. \quad (6.7)$$

The following two lemmas enable us to prove in two steps that the velocity is reversed according to the law described by (1.6). Lemma 6.3 shows that if ω has a Dirac mass at \bar{t} , then the left velocity at \bar{t} is outgoing; Lemma 6.4 shows indeed that (1.6) holds.

LEMMA 6.3. *If $\omega(\{\bar{t}\})$ is strictly positive, then $\dot{y}(\bar{t} - 0)$ is strictly negative.*

Proof. The idea of the proof is to find two successive times $t_{m-1} \leq t_m < \bar{t}$ for which we can write down an estimate on the discrete velocities, and then to use lemma 2.1 to perform a discrete integration and to obtain a contradiction. We must deal with the fact that \dot{y}_h does not converge uniformly to \dot{y} .

Without loss of generality, we may assume that \dot{y} is continuous on the right and that for all $h \leq h_1$, \dot{y}_h is also continuous from the right. According to Helly's theorem, there exists a countable set D such that

$$\dot{y}_h(t) \rightarrow \dot{y}(t), \quad \forall t \text{ such that } t - \bar{t} \in (\tau_{-5}, \tau_2) \setminus D.$$

Assume that $\dot{y}(\bar{t})$ vanishes; therefore, $\dot{y}(\bar{t} + 0)$ is strictly positive. Choose $\alpha = \dot{y}(\bar{t} + 0)/4$, and let τ_{-4} and τ_1 be such that

$$\begin{aligned} \tau_{-5} &\leq \tau_{-4} < \bar{t} < \tau_1 \leq \tau_2 \\ 6C_9(\tau_1 - \tau_{-4}) &\leq \alpha, \end{aligned} \quad (6.8)$$

and

$$\omega([t_0 + \tau_{-4}, \bar{t})) \leq \alpha, \quad \omega((\bar{t}, t_0 + \tau_1]) \leq \alpha. \quad (6.9)$$

An integration of (6.5) on appropriate intervals yields

$$\forall t \in (t_0 + \tau_{-4}, \bar{t}), \quad |\dot{y}(t \pm 0)| \leq \alpha + C_9(\bar{t} - t), \quad (6.10)$$

$$\forall t \in (\bar{t}, t_0 + \tau_1), \quad \dot{y}(t \pm 0) \geq \omega(\{\bar{t}\}) - \alpha - C_9(t - \bar{t}). \quad (6.11)$$

Choose $\tau_{-3} \in (\tau_{-4}, \bar{\tau}) \setminus D$ and $\tau_{-1} \in (\tau_{-3}, \bar{\tau}) \setminus D$; since ω_h is a nonnegative measure, we have the following inequality for all $\tau' \in (\tau_{-3}, \tau_{-1})$ and all $\tau'' \in (\tau', \tau_{-1})$:

$$\begin{aligned} |\dot{y}_h(t_0 + \tau') - \dot{y}_h(t_0 + \tau'')| &\leq \omega_h((t_0 + \tau', t_0 + \tau'')) + C_9(\tau'' - \tau' + h) \\ &\leq \omega_h([t_0 + \tau_{-3}, t_0 + \tau_{-1}]) + C_9(\tau'' - \tau' + h). \end{aligned}$$

We integrate $\omega_h - \omega$ on the interval $[t_0 + \tau_{-3}, t_0 + \tau_{-1}]$; since the measures ω and ω_h do not charge $t_0 + \tau_{-3}$ and $t_0 + \tau_{-1}$, we find that

$$\begin{aligned} \omega_h([t_0 + \tau_{-3}, t_0 + \tau_{-1}]) - \omega([t_0 + \tau_{-3}, t_0 + \tau_{-1}]) \\ = \dot{y}_h(t_0 + \tau_{-1}) - \dot{y}_h(t_0 + \tau_{-3}) - \dot{y}(t_0 + \tau_{-1}) + \dot{y}(t_0 + \tau_{-3}) \\ + \lambda([t_0 + \tau_{-3}, t_0 + \tau_{-1}]) - \lambda_h([t_0 + \tau_{-3}, t_0 + \tau_{-1}]), \end{aligned}$$

and therefore

$$\begin{aligned} \omega_h([t_0 + \tau_{-3}, t_0 + \tau_{-1}]) \\ \leq \omega([t_0 + \tau_{-3}, t_0 + \tau_{-1}]) + |\dot{y}_h(t_0 + \tau_{-1}) - \dot{y}(t_0 + \tau_{-1})| \\ + |\dot{y}_h(t_0 + \tau_{-3}) - \dot{y}(t_0 + \tau_{-3})| + C_9(2(\tau_{-1} - \tau_{-3}) + h). \end{aligned}$$

Choose now $\tau_{-2} \in (\tau_{-3}, \tau_{-1}) \setminus D$; then, for h small enough, $t_m = h \lfloor \tau_2/h \rfloor$ and $t_{m-1} = t_m - h$ belong to the interval (τ_{-3}, τ_{-1}) , and therefore,

$$|\dot{y}_h(t_m) - \dot{y}_h(t_{m-1})| \leq \alpha + C_9(2(\tau_{-1} - \tau_{-3}) + 3h) + \varepsilon_h, \quad (6.12)$$

where ε_h tends to 0 as h tends to 0. On the other hand, $\dot{y}_h(t_0 + \tau_{-2})$ tends to $\dot{y}(t_0 + \tau_{-2})$ and therefore, thanks to relation (6.10), there exists a family ε'_h such that

$$|\dot{y}_h(t_0 + \tau_{-2})| = |\dot{y}_h(t_m)| \leq \alpha + C_9(\bar{\tau} - \tau_{-2}) + \varepsilon'_h,$$

which is equivalent to

$$|\eta^m| \leq \alpha + C_9(\bar{\tau} - \tau_{-2}) + \varepsilon'_h; \quad (6.13)$$

we infer from (6.12) and (6.13) that

$$|\eta^{m-1}| \leq 2\alpha + C_9(2(\tau_{-1} - \tau_{-3}) + \bar{\tau} - \tau_{-2} + 3h) + \varepsilon_h + \varepsilon'_h.$$

Thus, for all $n \geq m$ we infer from Lemma 2.1 that

$$\begin{aligned} |\eta^m| &\leq 2\alpha + C_9(2(\tau_{-1} - \tau_{-3}) + 3h \\ &\quad + \bar{\tau} - \tau_{-2} + 2(t_m - t_m)) + \varepsilon_h + \varepsilon'_h. \end{aligned}$$

Therefore, in the limit, for all $t \geq t_0 + \tau_{-2}$

$$|\dot{y}(t)| \leq 2\alpha + C_9(2(\tau_{-1} - \tau_{-3}) + \bar{\tau} - \tau_{-2} + 2(t - \tau_{-2})).$$

and for all $t \in [t_0 + \tau_{-2}, t_0 + \tau_1]$

$$|\dot{y}(t)| \leq 2\alpha + C_9(2(\tau_{-1} - \tau_{-3}) + \bar{\tau} - \tau_{-2} + 2(\tau_1 - \tau_{-2})). \quad (6.14)$$

On the other hand, relation (6.11) implies that for all $t \in (\bar{t}, t_0 + \tau_1)$,

$$|\dot{y}(t)| \geq 3\alpha - C_9(\tau_1 - \bar{\tau}). \quad (6.15)$$

Under assumption (6.8), relation (6.15) contradicts relation (6.14). \square

We can conclude now the local study of the reflexion of the velocity by the following lemma:

LEMMA 6.4. *If $\omega(\{\bar{t}\})$ is strictly positive, then*

$$\dot{y}(\bar{t}) = -e\dot{y}(\bar{t} - 0). \quad (6.16)$$

Proof. Since $\dot{y}(\bar{t} - 0)$ is strictly negative, there exists a real number τ_{-3} such that $y(t)$ is strictly positive on $[t_0 + \tau_{-3}, \bar{t}) \subset [t_0 + \tau_{-5}, \bar{t})$. For all $\tau_{-2} \in (\tau_{-3}, \bar{\tau})$, there exists $\tau_{-1} \in (\tau_{-2}, \bar{\tau})$ and $h_1 > 0$ such that

$$\forall h \in (0, h_1], \quad \forall t \in [t_0 + \tau_{-2}, t_0 + \tau_{-1}), y_h(t) \geq \frac{y(t_0 + \tau_{-2})}{2}. \quad (6.17)$$

We prove now that there exists a maximal integer

$$m \in \{\lfloor \tau_{-3}/h \rfloor, \dots, \lfloor (\tau_0 + \varepsilon)/h \rfloor\}$$

such that

$$\forall l \in \{\lfloor \tau_{-3}/h \rfloor, \dots, m-1\}, \quad 2y^l - (1-e)y^{l-1} \geq 0, \quad (6.18)$$

and denoting

$$\sigma_h = t_{m-1} - t_0, \quad (6.19)$$

the time σ_h satisfies

$$\lim_{h \rightarrow 0} \sigma_h = \bar{\tau}. \quad (6.20)$$

Let us first observe that for all small enough h and all t_l belonging to $[t_0 + \tau_{-2}, t_0 + \tau_{-1}]$ we have

$$2y^l - (1-e)y^{l-1} \geq 0. \quad (6.21)$$

Indeed,

$$\begin{aligned} 2y^l - (1-e)y^{l-1} &= (1+e)y^l + (1-e)h\eta^{l-1} \\ &\geq \frac{1+e}{2}y(t_0 + \tau_{-2}) - h(1-e)\Lambda C_3, \end{aligned}$$

and if $2\Lambda C_3(1-e)h \leq (1+e)y(t_0 + \tau_{-2})$, we can see that (6.21) holds. Therefore m exists and

$$\liminf \sigma_h \geq \bar{\tau}.$$

On the other hand, if there existed $\tau_1 > \bar{\tau}$ such that for all $t_m \in [t_0 + \tau_{-3}, t_0 + \tau_1]$ we had (6.21), then ω_h would vanish on $(t_0 + \tau_{-3}, t_0 + \tau_1)$, which contradicts assumption (6.7). Therefore, we have shown that

$$\limsup \sigma_h \leq \bar{\tau},$$

i.e. (6.20). We integrate discretely equation (3.33), and we find that for $t \in [t_0 + \tau_{-3}, t_0 + \sigma_h]$

$$\begin{aligned} y_h(t) = & y_h(t_0 + \sigma_h) - (t_0 + \sigma_h - t)\dot{y}_h(t_0 + \sigma_h) \\ & + \int_t^{t_0 + \sigma_h} \lambda_h((s, t_0 + \sigma_h]) ds. \end{aligned} \quad (6.22)$$

In the limit we have,

$$y(t) = y(\bar{\tau}) - (\bar{\tau} - t) \lim_{h \downarrow 0} \dot{y}_h(t_0 + \sigma_h + 0) + \int_t^{t_0 + \bar{\tau}} \int_s^{t_0 + \bar{\tau}} \lambda(r) dr ds. \quad (6.23)$$

The comparison of (6.22) and (6.23) shows that

$$\lim_{h \downarrow 0} \dot{y}_h(t_0 + \sigma_h + 0) = \lim_{h \downarrow 0} \eta^{m-1} = \dot{y}(\bar{\tau} - 0). \quad (6.24)$$

Our purpose now is to obtain very precise estimates on the behavior of y_h beyond $t_0 + \sigma_h$. Thanks to the maximality of m , we have the relation

$$y^{m+1} = -ey^{m-1} + h^2\lambda^m; \quad (6.25)$$

let us estimate $2y^{m+1} - (1-e)y^m$: we substitute the value of y^{m+1} given by (6.25) into this expression, and we also use (3.34) with m replaced by $m-1$; we find

$$\begin{aligned} & 2y^{m+1} - (1-e)y^m \\ & = -[2y^{m-1} - (1-e)y^{m-2}] - (1-e)h^2\lambda^{m-1} + 2h^2\lambda^m. \end{aligned}$$

We apply relation (2.1) for $n = m+1$ and we find that

$$\begin{aligned} \eta^{m+1} + e\eta^{m-1} = & h(\lambda^{m+1} - \lambda^m) \\ & + \{-[2y^{m-1} - (1-e)y^{m-2}]h^{-1} - (1-e)h\lambda^{m-1} + 2h\lambda^m\}^+. \end{aligned}$$

Therefore, we have

$$\eta^{m+1} + e\eta^{m-1} \geq -2hC_9.$$

On the other hand, if $\xi = -[2y^{m-1} - (1-e)y^{m-2}]h^{-1} - (1-e)h\lambda^{m-1} + 2h\lambda^m$ is lesser than or equal to 0,

$$|\eta^{m+1} + e\eta^{m-1}| \leq 2hC_9;$$

if ξ is positive, then the sign condition on $2y^{m-1} - (1-e)y^{m-2}$ implies that

$$\eta^{m+1} + e\eta^{m-1} \leq h(\lambda^{m+1} + \lambda^m) - (1-e)h\lambda^{m-1}.$$

Thus, we have shown that

$$|\eta^{m+1} + e\eta^{m-1}| \leq 3C_9 h. \quad (6.26)$$

If e is strictly positive, then for all small enough h ,

$$\eta^{m+1} \geq e |\dot{y}(\bar{t} - 0)| / 2.$$

Let us estimate now the expression $2y^{m+2} - (1 - e)y^{m+1}$: we have

$$2y^{m+2} - (1 - e)y^{m+1} = -e[2y^m - (1 - e)y^{m-1}] + O(h^2).$$

If $2y^{m+2} - (1 - e)y^m$ is non-negative, then

$$y^{m+3} = 2y^{m+2} - y^{m+1} + h^2 \lambda^{m+2}.$$

We must estimate $2y^{m+3} - (1 - e)y^{m+2}$:

$$\begin{aligned} & 2y^{m+3} - (1 - e)y^{m+2} - 2y^{m+2} + (1 - e)y^{m+1} \\ &= h(2\eta^{m+2} - (1 - e)\eta^{m+1}) \\ &= h(1 + e)\eta^{m+1} + 2h^2 \lambda^{m+2}, \end{aligned}$$

and therefore $2y^{m+3} - (1 - e)y^{m+2}$ is non negative for all small enough h ; the repetition of the argument shows that there exists $\theta > 0$ such that for all small enough h and all $n \in \{m + 2, \dots, m + \lfloor \theta/h \rfloor\}$, the expression $2y^{m+1} - (1 - e)y^m$ is non negative, and thus we have the relations

$$y^m = y^{m+1} + h(n - m - 1)\eta^{m+1} + \sum_{j=m+2}^{m-1} (n - j)h^2 \lambda^j.$$

On the other hand, if $2y^{m+2} - (1 - e)y^m$ is negative, we must have

$$y^m = -\frac{(1 - e)h\eta^{m-1}}{1 + e} + O(h^2),$$

and therefore

$$y^{m-1} = -\frac{2h\eta^{m-1}}{1 + e} + O(h^2).$$

These relations and the assumption on the sign of $2y^{m+2} - (1 - e)y^m$ imply that

$$2y^{m+3} - (1 - e)y^{m+2} = -\frac{(4e^2 + e(1 - e)^2)h\eta^{m-1}}{1 + e} + O(h^2), \quad (6.27)$$

which is strictly positive for h small enough. But now, we can see that

$$y^{m+3} - y^{m+2} = -eh\eta^{m-1} + O(h^2),$$

which is strictly positive for small enough h , and therefore $2y^{m+4} - (1 - e)y^{m+3}$ is strictly positive for h small enough, since

$$2y^{m+4} - (1 - e)y^{m+3} \geq -he(1 + e)\eta^{m-1} + O(h^2);$$

the same argument as above shows now that there exists $\theta > 0$ such that for all $n \in \{m+3, \dots, m + \lfloor \theta/h \rfloor\}$,

$$y^m = y^{m+2} + h(n-m-2)\eta^{m+2} + \sum_{j=m+3}^{m-1} (n-j)h^2\lambda^j.$$

If we let $\sigma'_h = t_{m+1} - t_0$ in the first case and $\sigma'_h = t_{m+2} - t_0$ in the second case, we have now for $\sigma'_h \leq t - t_0 \leq \sigma'_h + \theta - h$

$$y_h(t) = y_h(t_0 + \sigma'_h) + (t - \sigma'_h - t_0)\dot{y}_h(t_0 + \sigma'_h) + \int_{t_0 + \sigma'_h}^t \lambda_h((s, t]) ds \quad (6.28)$$

and

$$y_h(t_0 + \sigma'_h) = O(h), \quad \dot{y}_h(t_0 + \sigma'_h) = -e\eta^{m-1} + O(h). \quad (6.29)$$

Passing to the limit in (6.28), we can see that

$$\dot{y}(\bar{t} + 0) = -e\dot{y}(\bar{t} - 0).$$

If we assume now that e vanishes, relation (6.26) implies

$$\eta^{m+1} = 0(h).$$

We observe that lemma 2.1 implies that for all n

$$|\eta^m| \leq |\eta^{m-1}| + 2C_9h,$$

which implies immediately that for $n \geq m+1$

$$|\eta^m| \leq |\eta^{m+1}| + 2hC_9(n-m-1),$$

which proves by a straightforward passage to the limit that

$$\dot{y}(\bar{t} + 0) = 0.$$

This completes the proof of the lemma. \square

7. Initial conditions. In this section we prove that the solution that we have constructed satisfies the initial conditions; we work under the hypotheses stated at the beginning of section 4.

LEMMA 7.1. *The function u satisfies the initial conditions*

$$u(0) = u_0, \quad p(0+0) = p_0.$$

Proof. By uniform convergence of u_h to u , it is clear that $u(0)$ is equal to u_0 . There remains to show that the initial condition on the impulsion is satisfied.

Assume first that u_0 belongs to the interior of K ; then there exist $h_1 > 0$ and $\tau_1 > 0$ such that for all $h \in (0, h_1]$ and for all $t \in [0, \tau_1]$

$$|u_h(t) - u_0| \leq \frac{1}{2} \inf\{|u_0 - y| : y \notin K\}.$$

Then for all t_m belonging to $(0, \tau_1]$, $(2U^m - (1 - e)U^{m-1} + h^2 F^m)/(1 + e)$ belongs to K for h small enough; we have indeed

$$\begin{aligned} & \left| \frac{2U^m - (1 - e)U^{m-1} + h^2 F^m}{1 + e} - u_0 \right| \\ & \leq \frac{1 - e}{1 + e} h C_1 + \frac{1}{2} \inf\{|u_0 - y| : y \notin K\} + \frac{h^2}{1 + e} C_8, \end{aligned}$$

which is strictly inferior to $\inf\{|u_0 - y| : y \notin K\}$ for h small enough. Thus the constraints are not saturated for $0 \leq t_m \leq \tau_1$ and the convergence is a classical result.

In the second case, u_0 belongs to ∂K ; we have taken admissible initial conditions, so that

$$\langle p_0, d\phi(u_0) \rangle_{u_0}^* \geq 0.$$

We use the construction and notations of section 3: Φ , Ψ , X^m , s^m , y^m and ζ^m have the same signification as there.

Taylor's formula yields

$$\zeta^0 = \frac{X^1 - X^0}{h} = D\Phi(u_0) \frac{U^1 - u_0}{h} + O(h),$$

and the definition (1.14) of U^1 gives

$$\zeta^0 = D\Phi(u_0)M(u_0)^{-1}p_0 + O(h). \quad (7.1)$$

Write

$$\begin{pmatrix} \sigma_0 \\ \eta_0 \end{pmatrix} = D\Phi(u_0)M(u_0)^{-1}p_0.$$

Then the normal and tangential components of the impulsion are given by

$$p_{0T} = M(u_0) \begin{pmatrix} \sigma_0 \\ 0 \end{pmatrix} \text{ and } p_{0N} = \eta_0 M(u_0)N(u_0).$$

We wish to prove

$$p(0 + 0) = p_0,$$

which is equivalent to

$$\dot{x}(0 + 0) = \begin{pmatrix} \sigma_0 \\ \eta_0 \end{pmatrix}.$$

We recall relation (3.33). Relation (7.1) implies that

$$\sigma^1 = (D\Phi(u_0)M(u_0)^{-1}p_0)' + O(h),$$

and together with (3.33), we obtain in the limit

$$\dot{s}(t) = (D\Phi(u_0)M(u_0)^{-1}p_0)' + O(t),$$

i. e.

$$\dot{s}(0+0) = \sigma_0.$$

Let us show now that

$$\dot{y}(0+0) = \eta_0,$$

considering two cases: $\eta_0 > 0$ and $\eta_0 = 0$. When η_0 vanishes, we have

$$y^1 = y^0 + h\eta_0 + O(h^2) = O(h^2),$$

and

$$\begin{aligned} y^2 &= -ey^0 + (2y^1 - (1-e)y^0)^+ + h^2\lambda^1 \\ &= 2(y^1)^+ + h^2\lambda^1 = O(h^2). \end{aligned}$$

Thus,

$$\eta^0 = O(h), \quad \eta^1 = O(h),$$

and relation (2.3) implies

$$|\eta^m| \leq O(h) + 2C_9h(n-1);$$

therefore, a passage to the limit gives immediately

$$\dot{y}(0+0) = 0.$$

If, on the other hand, η^0 is strictly positive, then

$$2y^1 - (1-e)y^0 = 2y^1 = 2h\eta^0 + O(h^2)$$

which is strictly positive if h is small enough. Let $\{1, \dots, m\}$ be the maximal interval such that

$$2y^n - (1-e)y^{n-1} > 0, \quad \text{if } n \leq m.$$

Then, for all $n \in \{1, \dots, m\}$,

$$\eta^n - \eta^{n-1} = h\lambda^n,$$

which implies by discrete integration that

$$\eta^n \geq \eta_0 - hnC_9,$$

as long as n belongs to $\{1, \dots, m\}$. Moreover, if we choose any $\tau_1 < \eta_0/(2C_9)$ and if n is at most equal to $\min(m, \lfloor \tau_1/h \rfloor)$, we can see that

$$y^m = y_0 + h(\eta^0 + \dots + \eta^{m-1}) \geq \frac{hn\eta_0}{2},$$

for all small enough values of h .

In particular, for all $n \leq \min(m, \lfloor \tau_1/h \rfloor)$,

$$2y^m - (1-e)y^{m-1} \geq \frac{(1+e)hn\eta_0}{2} - (1-e)h\Lambda C_3,$$

which proves that m is at least equal to $\lfloor \tau_1/h \rfloor$. Therefore, ω_h vanishes on the interval $(0, \tau_1 - h)$; in the limit, ω vanishes on $(0, \tau_1)$ and therefore

$$\dot{y}(0) = \eta_0,$$

which completes the proof of the lemma. \square

8. A priori estimates. In this section we prove that solutions of the problem (1.3), (1.4a), (1.4b), (1.4c), (1.5a), (1.5b), (1.8) and (1.9) satisfy an a priori estimate on an interval with non empty interior.

LEMMA 8.1. *Let R be strictly larger than $|p_0|_{u_0}^*$. Then there exists $\tau(R) > 0$ such that for all solution u of (1.3), (1.4a), (1.4b), (1.4c), (1.5a), (1.5b), (1.8) and (1.9) defined on $[t_0, t_0 + \tau]$, the following estimates hold:*

$$\forall t \in [t_0, t_0 + \min(\tau, \tau(R))], \quad |u(t) - u_0| \leq R, \quad |p(t)|_{u(t)}^* \leq R. \quad (8.1)$$

Proof. The measure λ appearing in (1.4b) can be decomposed in the sum of an atomic part λ_a and a diffuse part λ_d . At each point of the support of λ_a we have

$$|p(t+0)|_{u(t)}^* \leq |p(t-0)|_{u(t)}^* \quad (8.2)$$

thanks to relation (1.6). On any interval (t_1, t_2) which does not intersect the support of λ_a , we multiply relation (1.3) by \dot{u}^T on the left, and we find that

$$\frac{d}{dt} \frac{1}{2} \dot{u}^T M(u) \dot{u} = \dot{u}^T f(\cdot, u, p) + \frac{1}{2} \dot{u}^T (DM(u) \dot{u}) \dot{u}. \quad (8.3)$$

Define

$$E(u, p) = \frac{1}{2} \langle p, p \rangle_u^*, \quad z = |p|_u^*.$$

It is convenient to recall that

$$|p|_u^* = |M(u)^{-1/2} p| = |M(u)^{1/2} \dot{u}|.$$

Relations (8.2) and (8.3) imply that in the sense of measures

$$z \dot{z} = \dot{E} \leq \dot{u}^T f(\cdot, u, p) + \frac{1}{2} \dot{u}^T (DM(u) \dot{u}) \dot{u}. \quad (8.4)$$

Our purpose now is to transform (8.4) into a differential inequality. Let $\chi(u)$ be the norm of the bilinear mapping

$$(v_1, v_2) \mapsto M(u)^{-1/2} (DM(u) M(u)^{-1/2} v_1) M(u)^{-1/2} v_2.$$

With this definition,

$$|\dot{u}^T (DM(u) \dot{u}) \dot{u}| \leq \chi(u) z^3.$$

We write now

$$\begin{aligned} \dot{u}^T f(t, u, p) &= \dot{u}^T M(u)^{1/2} M(u)^{-1/2} f(t, u, p) \\ &= \dot{u}^T M(u)^{1/2} [M(u)^{-1/2} f(t, u, p) - M(u_0)^{-1/2} f(t, u_0, p_0) \\ &\quad + M(u_0)^{-1/2} f(t, u_0, p_0)]. \end{aligned}$$

Define

$$g(t) = |M(u_0)^{-1/2} f(t, u_0, p_0)|,$$

and let $\omega(\tau, R)$ be the Lipschitz constant of $(u, p) \mapsto M(u)^{-1/2}f(t, u, p)$ for $t \in [t_0, t_0 + \tau]$ and $\max(|u - u_0|, |p|_{u_0}^*) \leq R$; more precisely

$$\omega(\tau, R) = \sup \left\{ \frac{|M(u_1)^{-1/2}f(t, u_1, p_1) - M(u_2)^{-1/2}f(t, u_2, p_2)|}{|u_1 - u_2| + |p_1 - p_2|} : \right. \\ \left. t_0 \leq t \leq t_0 + \tau, \max(|u_1 - u_0|, |u_2 - u_0|, |p_1|_{u_1}^*, |p_2|_{u_2}^*) \leq R, \right. \\ \left. u_1 \neq u_2 \text{ or } p_1 \neq p_2 \right\}.$$

By construction, ω is continuous and it is an increasing function of τ and R .

Fix $R > |p_0|_{u_0}^*$.

If $t_0 \leq t \leq t_0 + \tau$ and if $\max(|u(t) - u_0|, |p(t)|_{u(t)}^*) \leq R$ on $[t_0, t_0 + \tau]$, we have the inequality

$$|\dot{u}^T f(\cdot, u, p)| \leq z(g + \omega(\tau, R)(|u - u_0| + |p - p_0|)).$$

But we can estimate $u(t) - u_0$:

$$|u(t) - u_0| \leq \int_{t_0}^t |\dot{u}(s)| ds \leq \int_{t_0}^t \|M(u)^{-1/2}\| z ds.$$

Therefore we have the estimate

$$|\dot{u}^T f(\cdot, u, p)| \leq zg + z\omega(\tau, R) \left(\int_{t_0}^t \|M(u)^{-1/2}\| z ds \right. \\ \left. + \|M(u)^{1/2}\| z + |p_0| \right),$$

and we conclude that z satisfies the differential inequality

$$\dot{z} \leq g + \omega(\tau, R) \left[\int_{t_0}^t \|M(u)^{-1/2}\| z ds + \|M(u)^{1/2}\| z + |p_0| \right] + \frac{1}{2} \chi(u) z^2.$$

Set

$$\alpha = \sup\{\|M(u)^{1/2}\| : |u - u_0| \leq R\}, \quad (8.5)$$

$$\beta = \sup\{\|M(u)^{-1/2}\| : |u - u_0| \leq R\}, \quad (8.6)$$

$$\gamma = 2 \sup\{\chi(u) : |u - u_0| \leq R\}.$$

While $t \leq t_0 + \tau$ and $\max(|u(t) - u_0|, |p(t)|_{u(t)}^*) \leq R$, z satisfies the following differential inequality

$$\dot{z} \leq g + \omega(\tau, R) \left[\beta \int_{t_0}^t z ds + \alpha z + |p_0| \right] + \gamma z^2. \quad (8.7)$$

Let ρ be any positive number; consider the integrodifferential equation

$$\dot{y} = g + \rho \left(\beta \int_{t_0}^t y ds + \alpha y + |p_0| \right) + \gamma |y|^2, \quad (8.8)$$

with the initial condition

$$y(t_0) = z(t_0).$$

It has a unique maximal solution which blows up in finite time, as soon as γ is strictly positive and $\rho|p_0| + \sup|g|$ is strictly positive. Let $\theta(\rho) \in [t_0, T]$ be the largest time for which

$$\forall t \in [t_0, \theta(\rho)], \quad y(t) \leq R, \quad \beta \int_{t_0}^t y \, ds \leq R.$$

As θ is a decreasing function of ρ , there exists a unique $\tau(R)$ such that

$$\theta(\omega(\tau(R), R)) = \tau(R).$$

Choose now

$$\rho = \omega(\tau(R), R).$$

Then we can compare the solution z of (8.7) and the solution y of (8.8), and we find immediately that

$$\forall t \in [t_0, t_0 + \min(\tau, \tau(R))], \quad z(t) \leq y(t). \quad (8.9)$$

This concludes the proof of the lemma. \square

9. Global results. We summarize the results obtained so far in the following Proposition:

PROPOSITION 9.1. *Assume that there exist strictly positive numbers τ , C_3 and $h_1 > 0$, and a sequence of solutions of the numerical scheme defined by (1.13), (1.14), (1.16) and (1.17), which satisfies the estimate (4.1). Then it is possible to extract from the sequence u_h defined by (5.1) a subsequence which converges to a solution of (1.3), (1.4a), (1.4b), (1.4c), (1.5a), (1.5b), (1.8) and (1.9). The convergence holds in the following sense: u_h converges uniformly to u_h on $[t_0, t_0 + \tau]$; \dot{u}_h converges to \dot{u} in $L^\infty(t_0, t_0 + \tau)$ weakly star and almost everywhere on $[t_0, t_0 + \tau]$, and \ddot{u}_h converges to \ddot{u} in the weak topology of measures. Moreover, for all $\tau \in (t_0, t_0 + \tau]$, we have the following convergence:*

$$\begin{aligned} \limsup_{h \downarrow 0} \sup \{ |V^m|_{U^m} : t_0 \leq t \leq t_0 + \tau \} \\ \leq \text{ess sup} \{ |\dot{u}(t)|_{u(t)} : t_0 \leq t \leq t_0 + \tau \}. \end{aligned} \quad (9.1)$$

Proof. The only statement which deserves a proof is the last one; if it is not true, there exists $\gamma > 0$, a sequence of time steps still denoted by h and a sequence of integers $m(h)$ such that

$$\left| V^{m(h)} \right|_{U^{m(h)}}^2 \geq \text{ess sup} \{ |\dot{u}(t)|_{u(t)}^2 : t_0 \leq t \leq t_0 + \tau \} + \gamma. \quad (9.2)$$

Without loss of generality, we may assume that $hm(h)$ tends to $\tau_2 \in [0, \tau]$.

First, τ_2 cannot be equal to 0: we have learnt in section 7 that there exists a constant C_{12} and a time τ_1 such that for all $h \leq h_1$ and all $m \leq \tau_1/h$,

$$|V^m - V^0| \leq C_{12}mh.$$

In particular, this estimate implies that

$$\left| V^{m(h)} \right|_{U^{m(h)}} = |v_0|_{u_0} + O(mh);$$

but $|v_0|_{u_0}$ is at most equal to $\{|\dot{u}(t)|_{u(t)} : t_0 \leq t \leq t_0 + \tau\}$, which contradicts (9.2). In the same fashion, we cannot have $u(t_0 + \tau_2) \in \text{int}(K)$; if it were the case, we could find an interval $[\tau_1, \tau_3]$ containing τ_2 and $h_1 > 0$ such that for all $h \in (0, h_1]$, $u_h([\tau_1, \tau_3])$ is included in a ball of radius r about $u(t_0 + \tau_2)$ included in the interior of K . But, in this case, \dot{u}_h converges uniformly to \dot{u} in $C^0([\tau_1, \tau_3])$ and this contradicts again (9.2).

Thus, we assume that τ_2 is strictly positive and that $u(t_0 + \tau_2)$ belongs to ∂K . Choose a coordinate system such that the origin is at $u(t_0 + \tau_2)$; let Ψ be the diffeomorphism defined at (3.13). In this case, $D\Psi(0)$ is given by (3.14). Define

$$\beta^m = (\xi^m)^T D\Psi(0)^T M(0) D\Psi(0) \xi^m.$$

Let us compare β^m to $|V^m|_{U^m}^2$; it is convenient to define

$$\tilde{V}^m = D\Psi(0) \xi^m;$$

then

$$\begin{aligned} |V^m|_{U^m}^2 - \beta^m &= (V^m)^T M(U^m) V^m - (\tilde{V}^m)^T M(0) \tilde{V}^m \\ &= (V^m)^T (M(U^m) - M(0)) V^m + (V^m - \tilde{V}^m)^T M(0) (V^m - \tilde{V}^m) \\ &\quad + 2(V^m - \tilde{V}^m)^T M(0) V^m. \end{aligned}$$

We observe that

$$|U^m| \leq \|u_h - u\| + C_3 |mh - \tau_1|, \quad \|X^m\| \leq \|U^m\|,$$

and that

$$|V^m - \tilde{V}^m| \leq C_4 \|\xi^m\| [2\Lambda \|X^m\| + \|X^m - X^{m-1}\|].$$

These observations enable us to estimate the difference: there exists a constant C_{13} such that

$$||V^m|_{U^m}^2 - \beta^m| \leq C_{13} (h + \|u - u_h\|_{C^0([t_0, t_0 + \tau])} + |mh - \tau_1|).$$

We infer from (3.43) that there exists a constant C_{14} such that

$$\beta^{m+1} \leq \min(\beta^m, \beta^{m-1}) + C_{14} h.$$

We use now (9.2): we can see that for all $m \leq m(h)$,

$$\beta^{m(h)} \leq \max(\beta^m, \beta^{m-1}) + C_{14} (m(h) - m),$$

so that

$$\begin{aligned} \max(|V^m|_{U^m}, |V^{m-1}|_{U^{m-1}}) &\geq \beta^{m(h)} - C_{14} (m(h) - m) h - \\ &\quad C_{13} (h + \|u - u_h\|_{C^0([t_0, t_0 + \tau])} + |mh - \tau_1|). \end{aligned}$$

If $\tau_4 < \tau_1$ is such that

$$\tau_1 - \tau_4 \leq \gamma / (4C_{14}),$$

and if

$$C_{13}(h + \|u - u_h\|_{C^0([t_0, t_0 + \tau])} + |mh - \tau_1|) \leq \gamma/(4C_{14}),$$

we can see that for all small enough h and all $m \in \{\lceil \tau_4/h \rceil, \dots, m(h)\}$ the following estimate holds:

$$\begin{aligned} & \max(|V^m|_{U^m}, |V^{m-1}|_{U^{m-1}}) \\ & \geq \text{ess sup}\{|\dot{u}(t)|_{u(t)}^2 : t_0 \leq t \leq t_0 + \tau\} + \gamma/2. \end{aligned} \quad (9.3)$$

But the function v_h defined by

$$v_h(t) = |V^m|_{U^m}^2 \text{ if } t \in [mh, (m+1)h)$$

converges almost everywhere on $[0, \tau]$ to $|\dot{u}(t)|_{u(t)}^2$; so does $\max(v_h(t-h), v_h(t))$. Therefore, in the limit, relation (9.3) leads to

$$\text{ess sup}_{t \in [t_0 + \tau_4, t_0 + \tau_2]} v_h(t) \geq \text{ess sup}\{|\dot{u}(t)|_{u(t)}^2 : t_0 \leq t \leq t_0 + \tau\} + \gamma/2,$$

which is a contradiction. \square

A corollary can be inferred immediately from this Proposition and Theorem 3.4:

COROLLARY 9.2. *For all admissible initial conditions u_0 and p_0 , there exists $\tau > 0$ and a solution of (1.3), (1.4a), (1.4b), (1.4c), (1.5a), (1.5b), (1.8) and (1.9) defined on $[t_0, t_0 + \tau]$.*

We have proved above the existence of a non-empty interval on which the numerical scheme converges to a solution of (1.3), (1.4a), (1.4b), (1.4c), (1.5a), (1.5b), (1.8) and (1.9). On the other hand, lemma 8.1 gives a priori estimates on the solution of such a problem.

We couple now the a priori estimates with the local convergence result to obtain a global result:

THEOREM 9.3. *Let R be strictly larger than $|p_0|_{u_0}^*$, and let $\tau(R)$ be given as in lemma 8.1. Then, for all small enough h , the solution U^m of the numerical scheme (1.13), (1.14), (1.16), (1.17) is defined on a discrete interval $\{0, \dots, m(h)\}$, such that*

$$hm(h) \rightarrow \tau(R);$$

moreover, the approximation u_h converges to a solution u of the continuous time equation, i.e. (1.3), (1.4a), (1.4b), (1.4c), (1.5a), (1.5b), (1.8) and (1.9), which is defined on $[t_0, t_0 + \tau(R)]$.

Proof. Let $\{0, \dots, m(h)\}$ be the discrete time interval for which the numerical scheme (1.13), (1.14), (1.16), (1.17) has a solution; we know from theorem 3.4 that

$$\liminf hm(h) = \tau > 0.$$

Assume that

$$\tau < \tau(R). \quad (9.4)$$

It is possible to extract from the sequence $(u_h)_h$ a subsequence, still denoted by u_h , such that on all subinterval $[0, \tau']$ included in $[0, \tau]$, u_h converges uniformly to u . In

particular, thanks to theorem 9.1 we will have

$$\begin{aligned} & \lim_{h \rightarrow 0} \max\{|V^m|_{U^m}^2 : 0 \leq m \leq \tau/h\} \\ & \leq \text{ess sup}\{|\dot{u}(t)|_{u(t)}^2 : t_0 \leq t \leq t_0 + \tau\}, \end{aligned}$$

and for h small enough we will have

$$\begin{aligned} & \lim_{h \rightarrow 0} \max\{|V^m|_{U^m}^2 : 0 \leq m \leq \tau/h\} \\ & \leq C_2 = 1 + R \max\{\|M(u)^{-1/2}\| : |u - u_0| \leq R\}. \end{aligned}$$

Thanks to theorem 3.5, we can find r_1 such that for all $\bar{u} \in K \cap B(u_0, R)$, for all U^{l-1} and U^l satisfying condition $E(\bar{u}, r_1, C_2, h)$, it is possible to define a solution of the scheme for $0 \leq (m-l)h \leq \tau$, where τ is independent of h . In particular, if we let $\tau' = \tau - \tau'/2$, $l = \lfloor \tau'/h \rfloor$ and $\bar{u} = u(\tau')$, we can extend the scheme up to m satisfying

$$mh \leq \min(lh + \tau'/2, \tau(R)),$$

which contradicts (9.4). This proves the desired result. \square

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